Sta 711: Homework 7

Fubini’s Theorem
1. Let $X \geq 0$ be a positive random variable and $\alpha > 0$. Show that

$$E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1}P(X > t)dt.\]

Note that the distribution $\mu(dx)$ of $X$ need not be absolutely continuous. Where did you use Fubini’s theorem?

Uniform Integrability (UI)
2. Fix $p > 0$ and set $X_n := n^p 1_{\{0 < \omega \leq 1/n\}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathbb{P} = \lambda$. Show explicitly that $\{X_n\}$ is UI for $p < 1$ and not for $p \geq 1$, by verifying that $E[X_n1_{\{X_n > t\}}]$ converges to zero uniformly as $t \to \infty$ for $p < 1$ and not for $p \geq 1$.

3. Let $\{X_n\}$ be an iid sequence of $L_1$ random variables and set $S_n = \sum_{i=1}^n X_i$. Show that the sequence of random variables $\{\bar{X}_n\}$ defined by $\bar{X}_n \equiv S_n/n$ is UI.

4. Let $X_n \sim \text{No}(0, \sigma_n^2)$. Find a simple condition on $\{\sigma_n^2\}$ such that $\{X_n\}$ is UI.

5. If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

6. Let $\phi(x) \geq 0$ be a nonnegative function which grows faster than $x$ at infinity, i.e., $\phi(x)/x \to \infty$ as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$E(\phi(|Z|)) \leq B.$$

Show that $\mathcal{C}$ is UI. Note: This implies any collection of random variables uniformly bounded in $L_p$ for some $p > 1$ is UI.

Convergence Theorems Revisited
7. Let $X$ be a non-negative real valued random variable. Show that:

(a) $\lim_{n \to \infty} nE(\frac{1}{X}1_{X > n}) = 0$.

(b) $\lim_{n \to \infty} n^{-1}E(\frac{1}{X}1_{X > n-1}) = 0$. 

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8. Let \( \{p_k\} \) be a probability mass function on \( \mathbb{N}_0 = \{0, 1, ...\} \) and define the generating function
\[
G(z) \equiv \sum_{k=0}^{\infty} p_k z^k \quad 0 \leq z \leq 1
\]

Use the Dominated Convergence Theorem to prove that
\[
\frac{d}{dz} G(z) = \sum_{k=0}^{\infty} p_k k z^{k-1} \quad 0 \leq z < 1. \tag{1}
\]

What is \( G(0) \)? \( G'(0) \)? \( G'(1) \)? How can you find each \( p_k \) explicitly from \( G(z) \)?

Extra Credit (+1pt):
Show that (1) also holds at \( z = 1 \), if we interpret \( \frac{d}{dz} G(z) \) as the left derivative
\[
G'(1-) \equiv \lim_{\epsilon \to 0} \frac{G(1) - G(1-\epsilon)}{\epsilon}.
\]