Sta 711: Homework 7

Fubini's Theorem

1. Let $X \geq 0$ be a positive random variable and $\alpha > 0$. Show that

$$\mathsf{E}(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} \mathsf{P}(X > t) dt.$$

Note that the distribution $\mu(dx)$ of X need not be absolutely continuous. Where did you use Fubini's theorem?

Uniform Integrability (UI)

2. Fix $p > 0$ and set $X_n := n^p \mathbf{1}_{\{0<\omega\leq 1/n\}}$ on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$. Show explicitly that $\{X_n\}$ is UI for $p < 1$ and not for $p \geq 1$, by verifying that $\mathsf{E}\big[X_n \mathbf{1}_{\{X_n > t\}}\big]$ converges to zero uniformly as $t \to \infty$ for $p < 1$ and not for $p \geq 1$.

3. Let $\{X_n\}$ be an iid sequence of L_1 random variables and set $S_n \equiv \sum_{i=1}^n X_i$. Show that the sequence of random variables $\{\bar{X}_n\}$ defined by $\bar{X}_n \equiv S_n/n$ is UI.

4. Let $X_n \sim \mathsf{No}(0, \sigma_n^2)$. Find a simple condition on $\{\sigma_n^2\}$ such that $\{X_n\}$ is UI.

5. If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

6. Let $\phi(x) \geq 0$ be a nonnegative function which grows faster than x at infinity, *i.e.*, $\phi(x)/x \to \infty$ as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$\mathsf{E}\big(\phi(|Z|)\big) \leq B.$$

Show that $\mathcal{C}$ is UI. Note: This implies any collection of random variables uniformly bounded in L_p for some $p > 1$ is UI.

Convergence Theorems Revisited

7. Let X be a non-negative real valued random variable. Show that:

   (a) $\lim_{n\to\infty} n\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n]}) = 0$.

   (b) $\lim_{n\to\infty} n^{-1}\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n^{-1}]}) = 0$.

8. Let $\{p_k\}$ be a probability mass function on $\mathbb{N}_0 = \{0, 1, ...\}$ and define the generating function

$$G(z) \equiv \sum_{k=0}^{\infty} p_k z^k \qquad 0 \leq z \leq 1$$

Use the Dominated Convergence Theorem to prove that

$$\frac{d}{dz} G(z) = \sum_{k=0}^{\infty} p_k k z^{k-1} \qquad 0 \leq z < 1. \tag{1}$$

What is $G(0)$? $G'(0)$? $G'(1)$? How can you find each p_k explicitly from $G(z)$?

Extra Credit (+1pt):

Show that (1) also holds at $z = 1$, if we interpret "$\frac{d}{dz} G(z)$" as the *left* derivative $G'(1-) \equiv \lim_{\epsilon \searrow 0} [G(1) - G(1-\epsilon)]/\epsilon$.