# Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2015 Nov 11, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible **simplify**.

Good luck!

Print Name Clearly: ______________________________

| | |
|---|---|
| 1. | /20 |
| 2. | /20 |
| 3. | /20 |
| 4. | /20 |
| 5. | /20 |
| Total: | /100 |

**Problem 1**: Let $\Omega := \mathbb{N} = \{1, 2, \ldots\}$ be the natural numbers. The sum $\zeta(p) := \Sigma_{n=1}^{\infty} \frac{1}{n^p}$ is infinite for $p \leq 1$ but finite for all $p > 1$. For $p = 4$ it is $\zeta(4) = \Sigma_{n=1}^{\infty}(1/n^4) = \pi^4/90$, so the set-function

$$\mathsf{P}[A] := \frac{90}{\pi^4} \sum_{n \in A} \frac{1}{n^4}$$

is a probability measure on the power set $\mathcal{F} = 2^\Omega$ of all subsets of $\Omega$. Define a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ by $X(\omega) \equiv \omega$.

a) (5) For $p > 0$, is $X \in L_p(\Omega, \mathcal{F}, \mathsf{P})$? If this depends on $p$, tell which $L_p$ spaces contain $X$. Choose one and give your reasoning:

◯ $X \in L_p$ for no $0 < p < \infty$ ◯ $X \in L_p$ for ____ $< p <$ ____

b) (5) Define a sequence of sequence of truncated approximations to $X$ by $X_n(\omega) \equiv \min(n, \omega)$. Are *they* in $L_p$? Why?

◯ $X_n \in L_p$ for no $0 < p < \infty$ ◯ $X_n \in L_p$ for ____ $< p <$ ____

c) (5) Does $X_n \to X$ almost-surely? ◯ Yes ◯ No Explain:

d) (5) Does $X_n \to X$ in $L_1$? Pick one: ◯ Yes ◯ No Explain:

**Problem 2**: Let $\{U_n\}_{n\in\mathbb{N}} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1]$ be independent uniformly-distributed random variables on the unit interval.

a) (5) What is the probability that at least one of the events

$$A_n := \{\omega : U_n(\omega) < \frac{1}{n+1}\}$$

occurs? ____ Why?

b) (5) What is the probability that *all* of the events

$$B_n := \{\omega : U_n(\omega) \le \exp\big(-2^{-n}\big)\}$$

occur? ____ Why?

c) (5) What is the probability that infinitely-many of the events

$$C_n := \{\omega : U_1(\omega) < U_2(\omega) < ... < U_n(\omega)\}$$

occur? ____ Why?

d) (5) Does the sequence of random variables $X_n := (\max_{1\le i\le n} U_i)^n$ converge to zero in probability? ◯ Yes ◯ No Why?

**Problem 3**: For two sequences of real numbers $\{a_n\} \subset \mathbb{R}$, $\{b_n\} \subset (0,1]$ and a sequence $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1]$ of independent uniform random on $(0,1]$, define random variables by

$$X_n := a_n \, \mathbf{1}_{(0,b_n]}(U_n) = \begin{cases} a_n & 0 < U_n \leq b_n \\ 0 & b_n < U_n \leq 1 \end{cases}$$

The random variables $\{X_n\}$ converge to zero in $L_1$ if and only if the sequences $\{a_n\}$ and $\{b_n\}$ satisfy the condition $|a_n| b_n \to 0$.

a) (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ in $L_2$?

b) (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ in $L_\infty$?

c) (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ *pr.*?

d) (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ *a.s.*?

**Problem 4**: Let $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ and set

$$X_n := -\log U_n \qquad S_n := \sum_{k=1}^{n} X_k \qquad Y_n := (1-U_n)/U_n \qquad T_n := \sum_{k=1}^{n} Y_k$$

a) (4) For $t > 0$, find:

$\mathsf{P}[X_n > t] =$ ______________ $\mathsf{P}[Y_n > t] =$ ______________

b) (4) Find the indicated moments (Page 8 may let you avoid integration):

$\mathsf{E}[X_n] =$ _______ $\mathsf{E}[X_n^2] =$ _______ $\mathsf{E}[Y_n] =$ _______ $\mathsf{E}[Y_n^2] =$ _______

c) (4) In what way and to what limit does $S_n/n$ converge as $n \to \infty$? Why?

d) (4) In what way and to what limit does $T_n/n$ converge as $n \to \infty$? Why?

e) (4) In what way and to what limit does $(S_n - n)/\sqrt{n}$ converge as $n \to \infty$? Why?

**Problem 5**: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think a question seems ambiguous or tricky.

a) T F If $X_n \to X$ *pr.* then $\mathsf{E}e^{itX_n} \to \mathsf{E}e^{itX}$ for all $t \in \mathbb{R}$.

b) T F For any $Y \in L_2$ and any $a > 0$, $\mathsf{P}[Y > a] \le \|Y\|_2^2/a^2$.

c) T F For any $Y \in L_2$, $\big(\mathsf{E}|Y|\big)^4 \le \big(\mathsf{E}Y^2\big)^2 \le \mathsf{E}Y^4$.

d) T F Two random variables $X$ and $Y$ are independent if and only if $\mathsf{E}[f(X \cdot Y)] = \mathsf{E}[f(X)]\,\mathsf{E}[f(Y)]$ for each bounded Borel function $f(x)$ on $\mathbb{R}$.

e) T F Two $\sigma$-algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are independent if and only if $\mathsf{E}[XY] = \mathsf{E}X \cdot \mathsf{E}Y$ for all random variables $X \in L_\infty(\Omega, \mathcal{G}, \mathsf{P})$, $Y \in L_\infty(\Omega, \mathcal{H}, \mathsf{P})$.

f) T F If $\mathsf{P}\big[|X_n - X| > \epsilon\big] \to 0$ for each $\epsilon > 0$, then $\mathsf{E}|X_n - X| \to 0$.

g) T F If $\mathsf{E}|X_n - X|^2 \to 0$, then $\mathsf{P}\big[|X_n - X| > \epsilon\big] \to 0$ for each $\epsilon > 0$.

h) T F If $\mathsf{E}|X_n - X|^2 \to 0$, then $\mathsf{E}|X_n - X| \to 0$.

i) T F If $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ are independent with the standard exponential distribution and $S_n := \sum_{i=1}^n X_i$, then $(S_n - n)/n$ converges to one *a.s.* as $n \to \infty$.

j) T F If $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ and $S_n := \sum_{i=1}^n X_i$, then $(S_n - n)/\sqrt{n}$ has approximately a $\mathsf{No}(0,1)$ distribution for large $n$.

**Blank Worksheet**

**Another Blank Worksheet**

| Name | Notation | pdf/pmf | Range | Mean $\mu$ | Variance $\sigma^2$ | |
|---|---|---|---|---|---|---|
| **Beta** | $\mathsf{Be}(\alpha,\beta)$ | $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | $x\in(0,1)$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |
| **Binomial** | $\mathsf{Bi}(n,p)$ | $f(x)=\binom{n}{x}p^x q^{(n-x)}$ | $x\in 0,\cdots,n$ | $n\,p$ | $n\,p\,q$ | $(q=1-p)$ |
| **Exponential** | $\mathsf{Ex}(\lambda)$ | $f(x)=\lambda\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $1/\lambda$ | $1/\lambda^2$ | |
| **Gamma** | $\mathsf{Ga}(\alpha,\lambda)$ | $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $\alpha/\lambda$ | $\alpha/\lambda^2$ | |
| **Geometric** | $\mathsf{Ge}(p)$ | $f(x)=p\,q^x$ | $x\in\mathbb{Z}_+$ | $q/p$ | $q/p^2$ | $(q=1-p)$ |
| | | $f(y)=p\,q^{y-1}$ | $y\in\{1,\ldots\}$ | $1/p$ | $q/p^2$ | $(y=x+1)$ |
| **HyperGeo.** | $\mathsf{HG}(n,A,B)$ | $f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$ | $x\in 0,\cdots,n$ | $n\,P$ | $n\,P\,(1-P)\frac{N-n}{N-1}$ | $(P=\frac{A}{A+B})$ |
| **Logistic** | $\mathsf{Lo}(\mu,\beta)$ | $f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\pi^2\beta^2/3$ | |
| **Log Normal** | $\mathsf{LN}(\mu,\sigma^2)$ | $f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}_+$ | $e^{\mu+\sigma^2/2}$ | $e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$ | |
| **Neg. Binom.** | $\mathsf{NB}(\alpha,p)$ | $f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$ | $x\in\mathbb{Z}_+$ | $\alpha q/p$ | $\alpha q/p^2$ | $(q=1-p)$ |
| | | $f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$ | $y\in\{\alpha,\ldots\}$ | $\alpha/p$ | $\alpha q/p^2$ | $(y=x+\alpha)$ |
| **Normal** | $\mathsf{No}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\sigma^2$ | |
| **Pareto** | $\mathsf{Pa}(\alpha,\epsilon)$ | $f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$ | $x\in\mathbb{R}_+$ | $\frac{\epsilon}{\alpha-1}{}^*$ | $\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$ | |
| | | $f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$ | $y\in(\epsilon,\infty)$ | $\frac{\epsilon\,\alpha}{\alpha-1}{}^*$ | $\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$ | $(y=x+\epsilon)$ |
| **Poisson** | $\mathsf{Po}(\lambda)$ | $f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$ | $x\in\mathbb{Z}_+$ | $\lambda$ | $\lambda$ | |
| **Snedecor $F$** | $F(\nu_1,\nu_2)$ | $f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$ | $x\in\mathbb{R}_+$ | $\frac{\nu_2}{\nu_2-2}{}^*$ | $\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}{}^*$ | |
| **Student $t$** | $t(\nu)$ | $f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$ | $x\in\mathbb{R}$ | $0^*$ | $\nu/(\nu-2)^*$ | |
| **Uniform** | $\mathsf{Un}(a,b)$ | $f(x)=\frac{1}{b-a}$ | $x\in(a,b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | |
| **Weibull** | $\mathsf{We}(\alpha,\beta)$ | $f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$ | $x\in\mathbb{R}_+$ | $\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$ | $\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$ | |