Independence

1. Let \( \{B_i\} \) be independent events. For \( n \in \mathbb{N} \) show that
\[
P \left( \bigcup_{i=1}^{n} B_i \right) = 1 - \prod_{i=1}^{n} [1 - P(B_i)] \geq 1 - \exp \left\{ - \sum_{i=1}^{n} P(B_i) \right\}
\]
and conclude that \( P \left[ \bigcup_{i=1}^{\infty} B_i \right] = 1 \) if each \( P[B_i] \geq \epsilon \) for some \( \epsilon > 0 \). Show that this conclusion would be false without the assumption of independence.

2. If \( \{A_n, n \in \mathbb{N}\} \) is a sequence of events such that \( P[A_n] = \frac{1}{3} \) for each \( n \) and
\[
(\forall n \neq m \in \mathbb{N}) \quad P(A_n \cap A_m) = P(A_n)P(A_m),
\]
does it follow that the events \( \{A_n\} \) are independent? Give a proof or counter-example. Note \( 1/3 \neq 1/2 \).

3. Show that a random variable \( Y \) is independent of itself if and only if, for some constant \( c \in \mathbb{R} \),
\[
P[Y = c] = 1.
\]
Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel measurable, and \( X \) any random variable. Can \( Y := f(X) \) and \( X \) be independent? Explain your answer.

4. Give an example to show that an event \( A \in \mathcal{F} \) may be independent of each \( B \) in some collection \( \mathcal{C} \subset \mathcal{F} \) of events, but not independent of \( \sigma(\mathcal{C}) \). Prove this is impossible if \( \mathcal{C} \) is a \( \pi \)-system (i.e., in that case \( A \) must be independent of \( \sigma(\mathcal{C}) \)).

5. Give a simple example to show that two random variables on the same space \( (\Omega, \mathcal{F}) \) may be independent according to one probability measure \( P_1 \) but dependent with respect to another \( P_2 \).

Fubini’s Theorem

6. Let \( X \geq 0 \) be a positive random variable and \( \alpha > 0 \). Show that
\[
E(X^\alpha) = \alpha \int_{0}^{\infty} t^{\alpha-1}P(X > t)dt.
\]
Note that the distribution \( \mu(dx) \) of \( X \) need not be absolutely continuous. Where did you use Fubini’s theorem?

7. Define measure spaces \( (\Omega_i, \mathcal{F}_i, \mu_i) \), for \( i = 1, 2 \) as follows. Let each \( \Omega_i := (0, 1] \), the unit interval, with \( \sigma \)-algebras
\[
\mathcal{F}_1 = \mathcal{B} = \text{Borel sets of (0,1]} \quad \mathcal{F}_2 = 2^{\Omega} = \text{All subsets of (0,1]},
\]
and let $\mu_1 = \lambda$ be Lebesgue measure and $\mu_2$ counting measure— so $\mu_1(A)$ is the length of any Borel set $A \in \mathcal{F}_1$ and $\mu_2(B)$ is the cardinality of $B \subset (0,1]$. Define
\[
 f(x, y) := 1_{x=y}(x, y)
\]
Set
\[
 I_1 := \int_{\Omega_1} \left[ \int_{\Omega_2} f(x,y) \mu_2(dy) \right] \mu_1(dx) \quad I_2 := \int_{\Omega_1} \left[ \int_{\Omega_2} f(x,y) \mu_1(dx) \right] \mu_2(dy)
\]
Compute $I_1$ and $I_2$. Is $I_1 = I_2$? Are the measures $\mu_1$ and $\mu_2$ $\sigma$-finite? Why doesn’t Fubini’s theorem hold here?

8. This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose $F$ and $G$ are two distribution functions with no common points of discontinuity on an interval $(a, b]$. Show that
\[
 \int_{(a,b]} G(x) F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)
\]
where “$G(dx)$” denotes the measure on $(\mathbb{R}, \mathcal{B})$ with DF $G(x)$. Show that the formula fails if $F$ and $G$ have common discontinuities.

Zero-One Laws

9. Let $\{X_n\}$ be a sequence of Bernoulli random variables with
\[
 P(X_n = 1) = n^{-p} \quad P(X_n = 0) = 1 - n^{-p}
\]
for some $p > 0$. For $p = 2$ show that the partial sum
\[
 S_n := \sum_{k=1}^{n} X_k
\]
converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, for which $p > 0$, does $S_n$ converge? Why?

10. Let $\{X_n\}$ be an iid sequence of random variables with a non-degenerate distribution (i.e., for some $B \in \mathcal{B}$, $0 < P[X_n \in B] < 1$). Show that
\[
 P[\omega : X_n(\omega) \text{ converges}] = 0
\]
11. Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$, there exist constants $c_n \to \infty$ such that
\[
 P \left( \lim_{n \to \infty} \frac{X_n}{c_n} = 0 \right) = 1.
\]
Give a careful description of how you choose $c_n$ (it will depend on the distributions of the $X_n$). Find a suitable sequence $\{c_n\}$ explicitly for an iid sequence $\{X_n\} \overset{\text{iid}}{\sim} \text{Ex}(1)$ of unit-rate exponentially-distributed random variables to ensure that $X_n/c_n \to 0$ almost surely.