STA 711: Note from Nov 13

Let \( \{X_n\} \) be independent random variables for \( n \in \mathbb{N} \) with

\[
P[X_n = x] = \begin{cases} 
    n^{-1} & x = 1 \\
    1 - n^{-1} & x = 0.
\end{cases}
\]

In what sense(s) does \( T_m := \sum_{1 \leq n \leq m} n^{-1} X_n \) converge to a finite random variable \( T \) as \( m \to \infty \)?

Convergence in \( L_1 \) to \( T := \sum_{1 \leq n < \infty} n^{-1} X_n \) is easy to show, either using the monotone convergence theorem and the calculation

\[
E[T] = \lim_{m \to \infty} E[T_m] = \sum_{n=1}^{\infty} E[X_n/n] = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty
\]

or the explicit bound

\[
E[T - T_m] = \sum_{m+1}^{\infty} \frac{1}{n^2} < \int_{m}^{\infty} \frac{1}{x^2} dx = \frac{1}{m} \to 0.
\]

It follows immediately that \( T_m \) converges almost-surely, because \( T \) is infinite on the set

\[
\mathcal{N} = \{ \omega : T_m(\omega) \text{ does not converge } \}.
\]

Since \( T \in L_1 \), necessarily \( P[\mathcal{N}] = 0 \), and \( T_m \to T \) a.s. (and so also pr.).

For any \( 1 \leq p < \infty \), Minkowski’s inequality and the calculation \( \|X_n\|_p = (1/n)^{1/p} = n^{-1/p} \) (so \( \|X_n/n\|_p = n'^{-1} \)) imply

\[
\|T - T_m\|_p = \left\| \sum_{m+1}^{\infty} X_n/n \right\|_p \leq \sum_{m+1}^{\infty} \|X_n/n\|_p \\
= \sum_{m+1}^{\infty} n'^{-1} \quad < \int_{m}^{\infty} x^{-1/p} dx = pm^{-1/p} \to 0,
\]

so also \( T_m \to T \) in \( L_p \) for all \( 1 \leq p < \infty \).

It doesn’t converge in \( L_\infty \), though, because for any \( B < \infty \) and any \( N \) large enough that \( \sum_{m < n \leq N} \frac{1}{n} > B \) (always possible since the harmonic series diverges),

\[
P[\|T - T_m\| > B] \geq P[X_n = 1 \text{ for } m < n \leq N] \\
\geq \prod_{m < n \leq N} \frac{1}{n} = \frac{m!}{N!} > 0,
\]

The smallest possible choice will be approximately \( N \approx me^B \).