Sta 711: Homework 4

Expectation

Feel free to use the result of one problem in your solution to a subsequent problem.

1. Let $X := (X_1, X_2)$ be distributed uniformly over the triangle in $\mathbb{R}^2$ with vertices $\{(−1, 0), (1, 0), (0, 1)\}$. Evaluate $E(X_1 + X_2)$ (no need to approximate by simple functions, just find the value).

2. Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ and, for $n \in \mathbb{N}$, set

$$X_n(\omega) := \min \left(2^n, 2\pi |2^n X(\omega)|\right)$$

Prove that $X_n$ is simple (how many values can it take on?) and $X_n \nrightarrow X$. Note you must show both monotonicity and convergence. For $\omega \in \Omega$ and $\epsilon > 0$, how big must $n$ be to ensure $|X - X_n| < \epsilon$?

3. Suppose $X \in L_1(\Omega, \mathcal{F}, P)$, i.e., $E|X| < \infty$. Show that\footnote{The “expectation of a random variable $X$ over an event $A$” can be written in many ways, including $\int_A X dP = E[X \mathbf{1}_A] = \int_A X(\omega) dP(\omega)$.}

$$\int_{|X| > m} X dP \to 0 \quad \text{as} \quad m \to \infty.$$ 

4. Let $\{A_n\}$ denote a sequence of events such that $P(A_n) \to 0$ as $n \to \infty$ and let $X \in L_1$. Show that

$$E[X \mathbf{1}_{A_n}] = \int_{A_n} X dP \to 0$$

Hint: Use problem 3.

5. Fix a probability space $(\Omega, \mathcal{F}, P)$ and define a distance measure $d$ on $\mathcal{F}$ by $d(A, B) \equiv P(A \Delta B)$ where (as usual) $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference. Show that, if $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $d(A_n, A) \to 0$, then

$$\int_{A_n} X dP \to \int_A X dP$$

for every $X \in L_1(\Omega, \mathcal{F}, P)$. Hint: Use problem 4.
Convergence Theorems

6. Let \( X \geq 0 \) be a non-negative random variable. Define sequences of random variables \( X_n \) and of extended real numbers \( 0 \leq S_n \leq \infty \) for positive integers \( n \in \mathbb{N} \) by:

\[
X_n = \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{\{k+1 \leq X \leq k+1\}} \quad S_n = \sum_{k=0}^{\infty} \frac{k}{2^n} P \left\{ \frac{k}{2^n} < X \leq \frac{k+1}{2^n} \right\}
\]

Is \( X_n \) “simple”? What is \( \lim_{n \to \infty} S_n \)? Justify your answers.

7. Define a sequence of random variables on \( (\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda) \) by

\[
X_n = \frac{n}{\log(n+1)} 1_{[0, \frac{1}{n}]} \quad n \in \mathbb{N}
\]

Show that \( P[X_n \to 0] = 1 \), and that \( \mathbb{E}(X_n) \to 0 \). Also show that the Dominated Convergence Theorem does not apply to this example. Why?

8. Let \( \{Y_n\} \) be a sequence of random variables for \( n \in \mathbb{N} \) with

\[
P(Y_n = \pm n^3) = \frac{1}{2n^2}, \quad P(Y_n = 0) = 1 - \frac{1}{n^2}
\]

One can (but you don’t have to) use the Borel-Cantelli lemma to show that \( Y_n \to 0 \) a.s. Compute \( \lim_{n \to \infty} \mathbb{E}(Y_n) \). Is the Dominated Convergence Theorem applicable? Why or why not?

9. Let \( \{X_n\}, X \) be random variables with \( 0 \leq X_n \to X \). If \( \sup_n \mathbb{E}(X_n) \leq K < \infty \), show that \( X \in L_1 \) and \( \mathbb{E}(X) \leq K \). Does \( X_n \to X \) in \( L_1 \)?

Domination

10. Let \( \{X_n\} \) be a sequence of random variables. Show that

\[
\mathbb{E} \left( \sup_{n \in \mathbb{N}} |X_n| \right) < \infty \tag{1a}
\]

if and only if there exists a random variable \( 0 \leq Y \in L_1 \) such that

\[
P(|X_n| \leq Y) = 1, \quad \forall n \in \mathbb{N}. \tag{1b}
\]

Thus, (1a) is exactly equivalent to domination in Lebesgue’s sense (but Lebesgue’s domination is often easier to verify).

11. Does the condition

\[
\sup_{n \in \mathbb{N}} \mathbb{E}(|X_n|) < \infty \tag{2}
\]

imply (1a)? Or is it implied by (1a)? For each direction (1a → 2 and 2 → 1a), give either a proof or a counter-example.