Sta 711: Homework 9

Uniform Integrability

1. True or false? Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.

   (a) If \( \{X_n, n \in \mathbb{N}\} \) is a uniformly integrable (UI) collection of random variables, then \( X_n \) is uniformly bounded in \( L_1 \).

   (b) Define a sequence \( \{X_n\} \) of random variables on the unit interval with Lebesgue measure, \((\Omega, \mathcal{F}, P)\) with \( \Omega = (0, 1] \), \( \mathcal{F} = \mathcal{B} \), and \( P = \lambda \), by \( X_n := \sqrt{n}1_{(0, \frac{1}{n}]} \). Then \( \{X_n\} \) is UI.

   (c) Let \( \{X_n\} \) be a sequence of random variables for which \( e^{X_n} \) is uniformly bounded in \( L_1 \), i.e., satisfies \( E|X_n| \leq B \) for some \( B < \infty \) and all \( n \). Then \( \{X_n\} \) is UI.

   (d) Let \( \{X_n\} \) be a sequence of random variables that is uniformly bounded in \( L_1 \), i.e., satisfies \( E|X_n| \leq B \) for some \( B < \infty \) and all \( n \). Then \( \{X_n\} \) is UI.

Characteristic Functions

2. Let \( X \) be a random variable, and define
   \[
   \phi_X(\theta) := E(e^{i\theta X}), \quad \theta \in \mathbb{R}
   \]
   Show that \( \phi_X(\theta) \) is uniformly continuous in \( \mathbb{R} \).

3. Find the characteristic functions of the following random variables:
   (a) \( W := c^1 \) (The superscripts in (a)-(c) are footnote indicators, not exponents)
   (b) \( X \sim \text{Un}(a, b)^2 \)
   (c) \( Y \sim \text{Ga}(\alpha, \lambda)^3 \)
   (d) \( Z_n = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \sim \text{Ga}(\alpha, \lambda) \)

   What is the distribution of \( Z_n \)? What happens as \( n \to \infty \)?

4. The distribution of a random variable \( X \) is called infinitely divisible if, for every \( n \in \mathbb{N} \), there exist \( n \) iid random variables \( \{Y_i\} \) such that \( X \) has the same distribution as \( \sum_{i=1}^n Y_i \). Use characteristic functions to show that if \( X \sim \text{Po}(\lambda) \), then \( X \) is infinitely divisible.\(^4\)

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\(^1\)A constant random variable with value \( c \in \mathbb{R} \)

\(^2\)Uniform, on the interval \((a, b) \subset \mathbb{R} \)

\(^3\)Gamma, with rate parameterization — with pdf \( f(y | \lambda) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y} / \Gamma(\alpha), y > 0. \)

\(^4\)Hint: If \( \{Y_i\} \) are independent with sum \( Y_+ := \sum Y_i \), then \( \phi_{Y_+}(\theta) = \prod \phi_{Y_i}(\theta) \) for all \( \theta \in \mathbb{R} \).
5. Suppose \( \{A_n, n \in \mathbb{N}\} \) are independent events satisfying \( \mathbb{P}(A_n) < 1, \forall n \in \mathbb{N} \). Show that \( \mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = 1 \) if and only if \( \mathbb{P}(A_n \text{ i.o.}) = 1 \) (“i.o.” means “infinitely often”, so the question concerns lim sup \( A_n \)). Give an example to show that the condition \( \mathbb{P}(A_n) < 1 \) cannot be dropped.

6. Let \( \{A_n\} \) be a sequence of events with \( \mathbb{P}(A_n) \to 1 \) as \( n \to \infty \). Prove that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( \mathbb{P}(\cap_k A_{n_k}) > 0 \).

7. Let \( A_n \) be a sequence of events that all satisfy \( \mathbb{P}(A_n) \geq \epsilon \) for some \( \epsilon > 0 \). Does there necessarily exist a subsequence \( \{n_k \to \infty\} \) with \( \mathbb{P}(\cap_k A_{n_k}) > 0 \)? Why or why not?

8. Let \( \{X_n\} \) be non-negative iid random variables, with tail \( \sigma \)-field

\[
\mathcal{T} := \bigcap_{n \in \mathbb{N}} \mathcal{F}_n', \quad \mathcal{F}_n' := \sigma\{X_m : m > n\}
\]

Is the event

\[
E = \{\text{There exists } \epsilon > 0 \text{ such that } X_n > n\epsilon \text{ for infinitely-many } n\}
\]

\[
= \bigcup_{\epsilon > 0} \bigcap_{n \geq 1} \bigcup_{m \geq n} \{\omega : X_m(\omega) > m\epsilon\}
\]

in \( \mathcal{T} \)? Prove or disprove it.

Express the probability \( \mathbb{P}[E] \) in terms of the random variables’ common distribution— for example, using their common CDF \( F(x) := \mathbb{P}[X_n \leq x] \) or moments \( \mathbb{E}[|X_n|^p] \) for some \( p > 0 \).