STA 711: Note from Nov 13

Let $\{X_n\}$ be independent random variables for $n \in \mathbb{N}$ with

$$\mathsf{P}[X_n = x] = \begin{cases} n^{-1} & x = 1 \\ 1 - n^{-1} & x = 0. \end{cases}$$

In what sense(s) does $T_m := \sum_{1 \le n \le m} n^{-1} X_n$ converge to a finite random variable T as $m \to \infty$?

Convergence in L_1 to $T := \sum_{1 \le n < \infty} n^{-1} X_n$ is easy to show, either using the monotone convergence theorem and the calculation

$$\mathsf{E}|T| = \sum_{n=1}^{\infty} \mathsf{E}[X_n/n] = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

or the explicit bound

$$\mathsf{E}|T - T_m| = \sum_{m+1}^{\infty} \frac{1}{n^2} < \int_m^{\infty} \frac{1}{x^2}\, dx = \frac{1}{m} \to 0.$$

It follows immediately that T_m converges almost-surely, because T is infinite on the set

$$\mathcal{N} = \{\omega : \ T_m(\omega) \text{ does not converge }\}.$$

Since $T \in L_1$, necessarily $\mathsf{P}[\mathcal{N}] = 0$, and $T_m \to T$ *a.s* (and so also *pr.*).

For any $1 \le p < \infty$, Minkowski's inequality and the calculation $\|X_n\|_p = (1/n)^{1/p} = n^{-1/p}$ (so $\|X_n/n\|_p = n^{-1-1/p}$) imply $T \in L_p$ and

$$\begin{aligned} \|T - T_m\|_p = \Big\| \sum_{m+1}^{\infty} X_n/n \Big\|_p &\le \sum_{m+1}^{\infty} \|X_n/n\|_p \\ &= \sum_{m+1}^{\infty} n^{-1-1/p} \quad < \int_m^{\infty} x^{-1-1/p}\, dx = pm^{-1/p} \to 0, \end{aligned}$$

so also $T_m \to T$ in L_p for all $1 \le p < \infty$.

It doesn't converge in L_∞, though, because for any $B < \infty$ and any N large enough that $\sum_{m < n \le N} \frac{1}{n} > B$ (always possible since the harmonic series diverges),

$$\begin{aligned} \mathsf{P}[|T - T_m| > B] &\ge \mathsf{P}[X_n = 1 \text{ for } m < n \le N] \\ &\ge \prod_{m < n \le N} \frac{1}{n} = \frac{m!}{N!} > 0. \end{aligned}$$

The smallest possible choice will be approximately $N \approx me^B$.