Final Examination

STA 711: Probability & Measure Theory

Sunday, 2014 Dec 14, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to [box] answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, or unreduced fractions. Wherever possible, Simplify! Good luck.

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Print Name: __________________________________________
Problem 1: Let \( \{X_n\} \) be independent random variables on some space \((\Omega, \mathcal{F}, P)\), not necessarily identically distributed or \(L_1\).

a) (5) If each \(X_n > 0\) a.s, is it possible to have \(\sum_{n=1}^{\infty} X_n < \infty\) a.s?
   ○ Yes  ○ No. If Yes, give an example; if No, say why.

b) (5) If \(X_n > 0\) and \(X_n \to 0\) a.s, does it follow that \(\sum_{n=1}^{\infty} X_n < \infty\) a.s?
   ○ Yes  ○ No. If No, give a counter-example; if Yes, say why.

c) (10) Set \(S_n := \sum_{i=1}^{n} X_i\). If \(\frac{1}{n} S_n \to Y\) a.s for some real-valued random variable \(Y\), use Kolmogorov’s zero-one law\(^1\) to prove that \(Y\) is almost-surely constant— i.e., that for some \(c \in \mathbb{R}\), \(P[Y = c] = 1\).

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\(^1\)Not his SLLN, which isn’t applicable here— note these \(\{X_n\}\) are not iid.
Problem 2: Let $X_n \rightarrow X$ a.s for some $X \in L_1(\Omega, \mathcal{F}, P)$, and let $Y \in L_2(\Omega, \mathcal{F}, P)$. For each part below answer “Yes” or “No” (4pts each).

If Yes, indicate which theorem best justifies your answer by selecting Fatou’s Lemma, Lebesgue’s Dominated or Monotone Convergence Theorems, the Borel/Cantelli lemma, Jensen’s Inequality, or the Minkowski, Hölder, or Markov Inequalities. No need to show work.

a) If $|X_n| \leq Y$, does $X_n \rightarrow X$ in $L_1$? ○ No ○ Yes, by:
○ Fat ○ DCT ○ MCT ○ B/C ○ Jen ○ Min ○ Höl ○ Mar

b) If $Y \leq X_n \nearrow X$, does $X_n \rightarrow X$ in $L_1$? ○ No ○ Yes, by:
○ Fat ○ DCT ○ MCT ○ B/C ○ Jen ○ Min ○ Höl ○ Mar

c) Is $E[X] \geq \lim \inf E[X_n]$? ○ No ○ Yes, by:
○ Fat ○ DCT ○ MCT ○ B/C ○ Jen ○ Min ○ Höl ○ Mar

d) If $X_n \leq Y$, is $\sum 1_{\{X_n > n\}}$ finite a.s? ○ No ○ Yes, by:
○ Fat ○ DCT ○ MCT ○ B/C ○ Jen ○ Min ○ Höl ○ Mar

e) If $X_n \searrow X \geq Y$, does $X_n \rightarrow X$ in $L_1$? ○ No ○ Yes, by:
○ Fat ○ DCT ○ MCT ○ B/C ○ Jen ○ Min ○ Höl ○ Mar

Fall 2014 2 Exam ends: 5pm
**Problem 3:** Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = 2^\Omega$ with uniform probability assignment $\mathbb{P}[E] := \#(E)/4$ for $E \in \mathcal{F}$, and consider events

$$A = \{a, b, c\} \quad B = \{b, c, d\} \quad C = \{a, d\} \quad D = \{c, d\}$$

a) (5) Which (if any) **pairs** of events from $\{A, B, C, D\}$ are independent?

b) (5) Find $\sigma(A, B)$, the smallest $\sigma$-algebra on $\Omega$ containing $A$ and $B$.

c) (5) Find $\sigma(A, B, C, D)$, the smallest $\sigma$-algebra on $\Omega$ containing each of the four events.

d) (5) Find $\pi(A, B, C, D)$, the smallest $\pi$-system on $\Omega$ containing each of the four events.
Problem 4:

a) (10) Recall that the pdf, CDF, and ch.f. of the normal $\mathcal{N}(\mu, \sigma^2)$ distribution are

\[
\begin{align*}
    f(x) &:= (2\pi\sigma^2)^{-1/2} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) = \frac{1}{\sigma} \varphi \left( \frac{x - \mu}{\sigma} \right) \\
    F(x) &:= \int_{-\infty}^{x} f(\xi) \, d\xi = \Phi \left( \frac{x - \mu}{\sigma} \right) \\
    \chi(\omega) &:= E \left[ e^{i\omega X} \right] = \exp \left( i\mu\omega - \frac{\sigma^2\omega^2}{2} \right)
\end{align*}
\]

Let $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$. If $\mu_n \to 1$ and $\sigma_n^2 \to 4$ as $n \to \infty$, prove that $X_n \Rightarrow \mathcal{N}(1, 2^2)$, i.e., that the random variables converge in distribution to the indicated normal.

b) (10) If $a_n \to 0$ is a sequence of real numbers and if $X \in L_1(\Omega, \mathcal{F}, P)$, prove that the sequence $Y_n := X + a_n$ converges in distribution to $X$. 

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Exam ends: 5pm
Problem 5: Let $\{A_n\}$ be independent events on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(A_n) = 1/n$ for $n \in \mathbb{N}$ and set

$$X_n := \sqrt{n}1_{A_n}.$$ 

a) (4) Is $\{X_n\}$ uniformly bounded in $L_1$? ○ Yes ○ No ○ Why?

b) (4) Is $\{X_n\}$ uniformly bounded in $L_2$? ○ Yes ○ No ○ Why?

c) (4) Is $\{X_n^2\}$ Uniformly Integrable? ○ Yes ○ No ○ Why?

d) (4) Find the covariance of $X_n$ and $X_m$ for all $n, m \in \mathbb{N}$.

e) (4) In which (if any) sense(s) does $X_n \rightarrow 0$ as $n \rightarrow \infty$?

○ a.s. ○ pr. ○ $L_1$ ○ $L_2$ ○ $L_\infty$ ○ in dist.
Problem 6: Let $X, Y \sim \text{No}(0, 1)$ and set $S := X + Y$, $Z := X - Y$, and $A := \{X < Y\} = \{Z < 0\}$.

(a) (8) Find the indicated covariances:

$$\text{Cov}(S, Z) = \text{________} \quad \text{Cov}(Y, Z) = \text{________}$$

(b) (12) Find the indicated conditional probabilities:

$$P[A \mid X] = \text{________} \quad P[A \mid S] = \text{________}$$
Problem 7: Again let $X, Y \sim \text{No}(0,1)$ and set $S := X + Y$, $Z := X - Y$, and $A := \{X < Y\} = \{Z < 0\}$.

a) (8) Find the indicated conditional expectations:

$$E[X \mid A] = \underline{\quad}, \quad E[S \mid A] = \underline{\quad}$$

b) (6) Find the indicated conditional expectations:

$$E[X \mid S] = \underline{\quad}, \quad E[Z \mid Y] = \underline{\quad}$$

c) (6) Find the indicated conditional expectations:

$$E[X \mid S, Z] = \underline{\quad}, \quad E[Z \mid X, S] = \underline{\quad}$$
Problem 8: Let $\Omega = (0,1)^2 = \{\omega = (\omega_1, \omega_2) : 0 < \omega_j \leq 1\}$ be the unit square, with the Borel sets $\mathcal{F} = \mathcal{B}(\Omega)$ and Lebesgue measure $P$ (i.e., area).

a) (5) Give an example\(^2\) of an event $A \in \mathcal{F}$ and random variable $W$ on $(\Omega, \mathcal{F}, P)$ that are independent and non-trivial—no constant RVs or null events.

b) (5) Give an example of a random variable $X$ on $(\Omega, \mathcal{F}, P)$ that is almost-surely finite but not bounded, if possible; if not, explain.

c) (5) Give an example $Y$ of a random variable in $L_1(\Omega, \mathcal{F}, P)$ that is not in $L_2(\Omega, \mathcal{F}, P)$, if possible; if not, explain.

d) (5) Give an example $Z$ of a random variable on $(\Omega, \mathcal{F}, P)$ whose distribution is neither continuous nor discrete, if possible; if not, explain.

\(^2\)You must be explicit—give $W(\omega)$ for every $\omega \in \Omega$, for example.
Problem 9: Let \( \{X_n\} \text{ iid} \sim \text{Ex}(1) \) be iid exponentially-distributed with mean 1, and set \( X^*_n := \max\{X_1, \ldots, X_n\} \) and \( S_n := \sum_{1 \leq j \leq n} X_j \).

a) (5) Show that \( P\left[X^*_n \leq (z + \log n)\right] \) converges as \( n \to \infty \) and find the limiting value \( G(z) := \lim_{n \to \infty} P[X^*_n \leq (z + \log n)] \) for every \( z \in \mathbb{R} \).

b) (5) Find the approximate median \( M_n \), so \( P[X^*_n \leq M_n] \approx 1/2 \).

c) (5) In which sense(s) does \( S_n/n \to 1? \)
   \( \bigcirc \) a.s.  \( \bigcirc \) pr.  \( \bigcirc \) \( L_1 \)  \( \bigcirc \) \( L_2 \)  \( \bigcirc \) \( L_\infty \)  \( \bigcirc \) in dist.

d) (5) Find the smallest bounds \( a_n > 0 \) and \( b_n > 0 \) you can to ensure
   \[ X^*_n \leq a_n S_n \quad S_n \leq b_n X^*_n \]
   \[ a_n = \quad \text{___________} \quad b_n = \quad \text{___________} \]
Problem 10: Let \( \{X, Y, Z\} \) be three RVs on the same probability space \( (\Omega, \mathcal{F}, P) \). Choose True or False below; no need to explain (unless you can’t resist). Each is 2pt.

a) T F If \( X \) is independent of \( Y \) and also \( X \) is independent of \( Z \) then \( X \) is independent of \( Y + Z \).

b) T F If \( P[X < Y < Z] = 1 \) then it’s not possible for \( X, Y, Z \) to be independent.

c) T F If \( X, Y, Z \) are independent and all are \( L_1 \) then the product \( XYZ \) is \( L_1 \) too.

d) T F If \( Z \) is measurable over \( \sigma(X, Y) \) then there exists some Borel function \( g \) on \( \mathbb{R}^2 \) such that \( Z = g(X, Y) \) a.s.

e) T F If \( X, Y \) are independent then the events \( A := [X < 0] \) and \( B := [Y > 0] \) are independent too.

f) T F If \( \sigma(X) \subset \sigma(Y) \) and \( X, Y \in L_1 \) then \( E[X \mid Y] \) is the constant random variable with value \( E[X] \).

g) T F If only finitely-many of the events \( A_n := [X > n] \) occur a.s., then \( X \in L_1(\Omega, \mathcal{F}, P) \).

h) T F If \( X, Y \in L_2 \) then \( XY \in L_1 \) with \( \|XY\|_1 \leq \|X\|_2 \|Y\|_2 \).

i) T F Let \( \phi(\omega) := E[e^{i\omega X}] \) and \( \psi(\omega) := E[e^{i\omega Y}] \) be the ch.f.s for \( X \) and \( Y \). Then the ch.f. for \( Z := X/Y \) is \( \phi(\omega)/\psi(\omega) \) if \( X, Y \) are independent.

j) T F If \( X \in L_1, Y \in L_2, \) and \( Z \in L_3 \), then \( (X + Y + Z) \in L_2 \).
Another Blank Worksheet
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<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
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<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0,1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha \beta}{(\alpha+\beta)^2 (\alpha+\beta+1)}$</td>
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<tr>
<td>Binomial</td>
<td>$\text{Bi}(n,p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$n p$</td>
<td>$n p q$</td>
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<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
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<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
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<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p q^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
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<tr>
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<td></td>
<td>$f(y) = p q^{y-1}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
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<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n,A,B)$</td>
<td>$f(x) = \frac{\binom{n}{x} (\frac{p}{1-p})^{n-x}}{A + B}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$n P$</td>
<td>$n P (1-P) \frac{N-n}{N-1}$</td>
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<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta \sqrt{2 \pi \beta^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\frac{\pi^2 \beta^2}{3}$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{x \sqrt{2 \pi \sigma^2}} e^{-\frac{\log(x-\mu)^2}{2 \sigma^2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2 \mu + \sigma^2} (e^{\sigma^2} - 1)$</td>
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<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{x-1}{x-\alpha} p^\alpha q^{x-\alpha}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
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<td>$f(y) = \binom{y-1}{y-\alpha} p^\alpha q^{y-\alpha}$</td>
<td>$y \in {\alpha, \ldots}$</td>
<td>$\alpha/p$</td>
<td>$\alpha q/p^2$</td>
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<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2 \pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2 \sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
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<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = \alpha e^{\alpha} / x^{\alpha+1}$</td>
<td>$x \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon \alpha}{\alpha-1}$</td>
<td>$\frac{\epsilon^2 \alpha}{(\alpha-1)^2 (\alpha-2)}$</td>
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<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
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<tr>
<td>Snedecor $F$</td>
<td>$\text{F}(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \times \frac{\nu_2}{\nu_2-2} \left[1 + \frac{\nu_2}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{\nu_1+\nu_2-2}{\nu_1</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{1}{\sqrt{\pi \nu}} \frac{1}{\sqrt{1+x^2/\nu}}^{(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu/(\nu-2)$</td>
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<tr>
<td>Uniform</td>
<td>$\text{Un}(a,b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a,b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\Gamma(1+\alpha)}{\beta^{\alpha}}$</td>
<td>$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$</td>
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