This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible simplify.

Good luck.

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<tr>
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Print Name: ______________________________
Problem 1: Let $Z \sim \mathcal{N}(0,1)$ and set $X := Z^2$, $\mathcal{G} := \sigma(Z)$, $\mathcal{H} := \sigma(X)$.

a) (5) Find $a, b \in \mathbb{R}$ such that the random variable $Y := a + bZ$ is the orthogonal projection of $X$ onto the span of $Z$, i.e., satisfies

$$\mathbb{E}(X - Y)Z = 0 \text{ and } \mathbb{E}(X - Y)1 = 0$$

$$Y = \underbrace{a}_{a} + \underbrace{b}_{b} Z$$

b) (5) Find the conditional expectation of $X$, given $\mathcal{G} = \sigma(Z)$:

$$\mathbb{E}[X \mid \mathcal{G}] = \text{________________________}$$

c) (5) Find the conditional expectation of $Z$, given $\mathcal{H} = \sigma(X)$:

$$\mathbb{E}[Z \mid \mathcal{H}] = \text{________________________}$$

d) (5) Give an event $A \in \mathcal{G} = \sigma(Z)$ that is not in $\mathcal{H} = \sigma(X)$, if possible, and an event $B \in \mathcal{H}$ that is not in $\mathcal{G}$, if possible. If not possible, explain why.

$$\mathcal{G} \setminus \mathcal{H} \ni A = \text{________________________} \text{ and } \mathcal{H} \setminus \mathcal{G} \ni B = \text{________________________}$$
Problem 2: Let $X_n \to X$ pr. for some $X \in L_1(\Omega, \mathcal{F}, P)$, and let $Y \in L_2(\Omega, \mathcal{F}, P)$. For each part below answer “Yes” or “No”.

If Yes, indicate which theorem best justifies your answer by selecting Fatou’s Lemma, Lebesgue’s Dominated or Monotone Convergence Theorems, the Borel/Cantelli lemma, Fubini’s Theorem, or the inequalities of Jensen, Minkowski, Hölder, or Markov. No need to show work.

a) If $|X_n|^{1/2} \leq Y \text{ a.s.}$, does $X_n \to X$ in $L_1$?  
   ○ No  ○ Yes, by:
   ○ Fat  ○ DCT  ○ MCT  ○ B/C  ○ Fub  ○ Jen  ○ Min  ○ Höl  ○ Mar

b) If $X_n \not\Rightarrow \ X \leq Y$, does $X_n \to X$ in $L_1$?  
   ○ No  ○ Yes, by:
   ○ Fat  ○ DCT  ○ MCT  ○ B/C  ○ Fub  ○ Jen  ○ Min  ○ Höl  ○ Mar

c) If $X_n \leq Y$, is $E[X] \geq \limsup E[X_n]$?  
   ○ No  ○ Yes, by:
   ○ Fat  ○ DCT  ○ MCT  ○ B/C  ○ Fub  ○ Jen  ○ Min  ○ Höl  ○ Mar

d) If $\sum_n 1_{\{X^2 \geq n\}} < \infty \text{ a.s.}$, is $X \in L_2$?  
   ○ No  ○ Yes, by:
   ○ Fat  ○ DCT  ○ MCT  ○ B/C  ○ Fub  ○ Jen  ○ Min  ○ Höl  ○ Mar

e) If $(\forall \epsilon > 0) \sum P(|X_n| > \epsilon) < \infty$, does $X_n \to 0 \text{ a.s.}$?  
   ○ No  ○ Yes, by:
   ○ Fat  ○ DCT  ○ MCT  ○ B/C  ○ Fub  ○ Jen  ○ Min  ○ Höl  ○ Mar
Problem 3: Let \( \Omega := \{0, 1, 2, 3\} \) with probability assignment \( P[E] := \sum_{\omega \in E} 2^{\omega}/15 \) for \( E \in \mathcal{F} := 2^\Omega \). Consider events \( A := \{0, 1\} \) and \( B := \{0, 2\} \), and random variables

\[ W(\omega) = \omega \quad X(\omega) = 2^\omega \quad Y(\omega) = 1_A(\omega) \quad Z(\omega) = 1_B(\omega) \]

a) (5) Find the expectation of each RV:

\[ E_W = \quad \quad E_X = \quad \quad E_Y = \quad \quad E_Z = \quad \quad \]

b) (5) Are \( \sigma(Y) \) and \( \sigma(Z) \) independent? \( \bigcirc \) Yes \( \bigcirc \) No Why?

c) (5) How many events are in the \( \sigma \)-algebra \( \sigma(Y, Z) \) generated by \( Y \) and \( Z \)? You need not enumerate them.

d) (5) Find the conditional expectation \( E[W \mid Y] \).
Problem 4: Let \( \{X_n\} \) and \( Y \) be real-valued random variables on \((\Omega, \mathcal{F}, P)\) and for \( n, k \in \mathbb{N} \) set \( A_{n,k} := \{ \omega : |X_n(\omega) - Y(\omega)| > \frac{1}{k} \} \).

a) (5) Give the exact conditions on \( A_{n,k} \) for \( X_n \to Y \) a.s.

b) (5) Give the exact conditions on \( A_{n,k} \) for \( X_n \to Y \) pr.

c) (5) Use your expressions above to prove that almost-sure convergence implies convergence in probability.

d) (5) Prove that \( \sin(X_n) \to \sin(Y) \) in \( L_1(\Omega, \mathcal{F}, P) \) if \( X_n \to Y \) a.s.
Problem 5: Let $\Omega = (0, 1]^2 = \{ (\omega_1, \omega_2) : 0 < \omega_j \leq 1 \}$ with Lebesgue measure $P$ on the Borel sets $\mathcal{F}$, and consider the random variables

$$X(\omega) := \omega_1 \quad Y(\omega) := \omega_2 \quad R(\omega) := (\omega_1^2 + \omega_2^2)^{1/2} \quad \Theta(\omega) := \arctan \omega_2 / \omega_1$$

(so $\omega_2 / \omega_1 = \tan \Theta$, with $0 < \Theta \leq \pi / 2$)

a) (4) Sketch an event $A \in \sigma(R)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(\Theta)$.

b) (4) Sketch an event $B \in \sigma(\Theta)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(R)$. 

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Problem 5 (cont’d): Recall $\Omega = (0, 1]^2$ with Lebesgue measure

c) (4) Sketch and label independent
   events $D \in \sigma(R)$ and $E \in \sigma(\Theta)$
   that are non-trivial—i.e., have
   probabilities $0 < P(D), P(E) < 1$.

XC) (+2) Are $\sigma(R)$ and $\sigma(\Theta)$ independent?  ○ Yes  ○ No  Why?

d) (8) Sketch and label the events
   $F := \{\omega : 0 < \omega_1 \leq \omega_2 \leq 1\}$ and
   $G := \{\omega : 0 < \omega_1 \leq \frac{1}{2}, 0 < \omega_2 \leq 1\}$,
   and compute $E[1_F \mid 1_G]$. No need
   to prove anything, just compute
   the conditional expectation.

$$E[1_F \mid 1_G] =$$
Problem 6: Let \( \{A_n\} \) be independent events on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \mathbb{P}(A_n) = 2^{-n} \) for \( n \in \mathbb{N} \). Fix \( \alpha > 0 \) and set

\[ X_n := \alpha^n 1_{A_n}. \]

In d) and e), “converge” means “converge to some finite random variable”.

a) (4) For which (if any) \( \alpha > 0 \) is \( \{X_n\} \) uniformly bounded in \( L_1 \)? Why?

b) (4) For which (if any) \( \alpha > 0 \) is \( \{X_n\} \) uniformly bounded in \( L_4 \)? Why?

c) (4) For which (if any) \( \alpha > 0 \) is \( \{X_n\} \) uniformly bounded in \( L_\infty \)? Why?

d) (4) For which (if any) \( \alpha > 0 \) does \( \sum_{n \in \mathbb{N}} X_n \) converge \( a.s.? \) Why?

e) (4) For which (if any) \( \alpha > 0 \) does \( \sum_{n \in \mathbb{N}} X_n \) converge in \( L_1 \)? Why?
Problem 7: Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be $\sigma$-fields on the set $\Omega = (0, 1]$, and let $Y$ be a $\mathcal{G}$-measurable random variable.

a) (10) Does it follow that $Y$ is also $\mathcal{H}$-measurable? ☐ Yes ☐ No
If “Yes”, give a proof; if “No”, give a counter-example\(^1\).

b) (10) Does it follow that $Y$ is also $\mathcal{F}$-measurable? ☐ Yes ☐ No
If “Yes”, give a proof; if “No”, give a counter-example\(^1\).

\(^1\)For counter-examples, you might use the $\sigma$-algebras $\mathcal{F}_n := \sigma\{(0, i/2^n] : 0 \leq i \leq 2^n\}$
Problem 8: Let \( \{X,Y,Z\} \) and \( \{X_n\} \) be RVs on the same probability space \((\Omega, \mathcal{F}, P)\). Choose True or False below; no need to explain (unless you can’t resist). Each is 2pt.

a) \( \text{T F} \) If \( E[\exp(X)] < \infty \) then also \( E|X| < \infty \).

b) \( \text{T F} \) If \( X_n \to X \) pr. and if \( \Omega \) is countable then \( X_n \to X \) a.s.

c) \( \text{T F} \) If \( X, Y \) are independent and both are in \( L_2 \) then the product \( XY \) is in \( L_2 \) too.

d) \( \text{T F} \) If \( Z = g(X,Y) \) for some Borel function \( g : \mathbb{R}^2 \to \mathbb{R} \) then \( \sigma(Z) \subseteq \sigma(X,Y) \).

e) \( \text{T F} \) If the events \( A := [X < x] \) and \( B := [Y > y] \) are independent for each \( x, y \in \mathbb{R} \) then \( X, Y \) are independent RVs.

f) \( \text{T F} \) If \( \sigma(X) \perp \sigma(Y) \) and \( X, Y \in L_1 \) then \( E[X \mid Y] \) is the constant random variable with value \( E[X] \).

g) \( \text{T F} \) If the sum \( \sum_n P(A_n) < \infty \) for the events \( A_n := [|X| > n] \), then \( X \in L_1(\Omega, \mathcal{F}, P) \).

h) \( \text{T F} \) If \( X, Y \in L_6 \) then \( XY \in L_3 \) with \( \|XY\|_3 \leq \|X\|_6 \|Y\|_6 \).

i) \( \text{T F} \) Let \( \phi(\omega) := E[e^{i\omega X}] \) and \( \psi(\omega) := E[e^{i\omega Y}] \) be the ch.f.s for \( X \) and \( Y \). Then the ch.f. for \( Z := X - Y \) is \( \phi(\omega)/\psi(\omega) \) if \( X, Y \) are independent.

j) \( \text{T F} \) If \( P[|X| > t] \geq P[|Y| > t] \) for each \( t > 0 \) then \( \|X\|_2 \geq \|Y\|_2 \).

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Another Blank Worksheet
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<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
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<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha^{3}}{(\alpha+\beta)^{2}(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in {0, \cdots, n}$</td>
<td>$np$</td>
<td>$npq$ ($q = 1-p$)</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = pq^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$ ($q = 1-p$)</td>
</tr>
<tr>
<td>HyperGeo</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(x) = \binom{n}{x} \frac{(\frac{A}{B})^x}{\Gamma(n+1)}$</td>
<td>$x \in {0, \cdots, n}$</td>
<td>$n$</td>
<td>$nP \left(1-P\right)^{n-n} \frac{A}{A+B}$ ($P = \frac{A}{A+B}$)</td>
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<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-\frac{(x-\mu)^2}{\beta}}}{\sqrt{2\pi\beta}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\frac{\beta^2}{3}$</td>
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<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$</td>
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<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = (x+\alpha-1) p^\alpha q^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$ ($q = 1-p$)</td>
</tr>
<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = (\epsilon/\alpha)(1 + x/\epsilon)^{-\alpha-1}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\epsilon}{\alpha - 1}$ *</td>
<td>$\frac{\epsilon^{2\alpha}}{(\alpha-1)^{2}(\alpha-2)} *$</td>
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<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\epsilon/\alpha - 1$ *</td>
<td>$\frac{\epsilon^{2\alpha}}{(\alpha-1)^{2}(\alpha-2)} *$ ($y = x + \epsilon$)</td>
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<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\pi \nu}} \left[ 1 + \frac{x^2}{\nu} \right]^{-\frac{\nu+1}{2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0^*$</td>
<td>$\nu/(\nu-2)^*$</td>
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<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
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<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\Gamma(1+\alpha^{-1})/\beta^{1/\alpha}$</td>
<td>$\Gamma(1+2/\alpha)/\beta^{2/\alpha}$</td>
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