Final Examination

STA 711: Probability & Measure Theory

Saturday, 2017 Dec 16, 7:00 – 10:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to [box] answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible simplify.

Good luck.

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<td>1.</td>
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Print Name: ________________________________
Problem 1: Let $\xi_1, \xi_2, \ldots$ be iid random variables with the $\text{Ex}(1/2)$ distribution (hence mean $E[\xi_j] = 2$... see distribution reference sheet, p. 15).

a) (8) Find non-random $a_n \in \mathbb{R}, b_n > 0$ such that $S_n := \sum_{1 \leq j \leq n} \xi_j$ satisfies

$$P[(S_n - a_n)/b_n \leq x] \to F(x)$$

for a non-trivial df $F$ (i.e., one for a distribution not concentrated at a single point). Give $a_n, b_n,$ and $F.$ Justify your answer.
Problem 1 (cont’d): Still \( \{\xi_j\} \overset{iid}{\sim} \text{Ex}(1/2) \).

b) (6) Find non-random \( a_n \in \mathbb{R}, b_n > 0 \) such that \( X_n := \min_{1 \leq j \leq n} \xi_j \) satisfies:
\[
P\left[ \frac{(X_n - a_n)}{b_n} \leq x \right] \rightarrow G(x)
\]
for a non-trivial df \( G \). Give \( a_n, b_n, \) and \( G \). Justify your answer.

c) (6) Find non-random \( a_n \in \mathbb{R}, b_n > 0 \) such that \( Y_n := \max_{1 \leq j \leq n} \xi_j \) satisfies
\[
P\left[ \frac{(Y_n - a_n)}{b_n} \leq x \right] \rightarrow H(x)
\]
for a non-trivial df \( H \). Give \( a_n, b_n, \) and \( H \). Justify your answer.
Problem 2: Let $\{X_n\}$ and $Y$ be real-valued random variables on $(\Omega, \mathcal{F}, P)$ such that $X_n \to Y$ (pr).

a) (10) Set $A_n := \{\omega: |X_n(\omega)| > n\}$. Prove that $P(A_n) \to 0$ as $n \to \infty$.

b) (10) Prove that $\exp(-X_n^2) \to \exp(-Y^2)$ in $L_1(\Omega, \mathcal{F}, P)$. 

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Problem 3: For each part below, select “True” or “False” and sketch a short explanation or counter-example to support your answer:

a) (4) T F If \( \{X_j\} \) are \( L_1 \) random variables and \( \sum \|X_j\|_1 < \infty \) then \( S_n := \sum_{1 \leq j \leq n} X_j \) converges in \( L_1 \) to a limit \( S \in L_1(\Omega, \mathcal{F}, \mathbb{P}) \).

b) (4) T F If \( \{X_n\} \), \( Y \) are \( L_2 \) random variables with \( \|X_n\|_2 \leq 10 \) and if \( X_n \to Y \) in \( L_1 \) then \( \mathbb{P}[X_n \to Y] = 1 \).

c) (4) T F If \( \{X_n\} \), \( Y \) are \( L_1 \) random variables with \( X_1 \leq 42 \) a.s. and if \( X_n \downarrow Y \) decreases to \( Y \) a.s., then \( X_n \to Y \) in \( L_1 \).

d) (4) T F If \( \{X_j\} \) are independent \( L_1 \) random variables with zero mean \( \mathbb{E}[X_j] = 0 \) then \( Y_n := 1 + \sum_{1 \leq j \leq n} j^2 X_j \) is a martingale.

e) (4) T F If \( X \in L_p \) for every \( 0 < p < \infty \) then also \( X \in L_\infty \), because \( \|X\|_p \to \|X\|_\infty \) as \( p \to \infty \).
Problem 4: Let $\Omega = \mathbb{N} = \{1, 2, \ldots\}$ be the natural numbers, with probability measure
\[ P(A) := \frac{6}{\pi^2} \sum_{\omega \in A} \frac{1}{\omega^2} \]
on the power set $A \in \mathcal{F} := 2^\Omega$. Note $P(\Omega) = 1$ because $\sum_{\omega=1}^\infty \frac{1}{\omega^2} = \pi^2/6$.

a) (4) Let $E := \{2j : j \in \mathbb{N}\}$ be the even numbers, $D := \{2^j : j \in \mathbb{N}\}$ the integer powers of two that are $\geq 2$, and $S := \{j^2 : j \in \mathbb{N}\}$ the squares. How many events are in each of the following classes? (Events $A \subseteq \Omega$, not elements $\omega \in A$)

\[ \sigma(E, S) : \quad \sigma(D, E) : \quad \pi(D, E) : \quad \sigma(D, E, S) : \]

b) (8) Find the indicated probabilities:

\[ P(E) = \quad P(D) = \]
Problem 4 (cont’d): Still $\Omega = \mathbb{N}$ and $P(A) := \frac{6}{\pi^2} \sum_{\omega \in A} \frac{1}{\omega^2}$. 

c) (8) Set $X(\omega) = 1_{\{\omega \leq 3\}}$. Find:

$$E(X \mid \sigma(E)) =$$
Problem 5: Let $X_j \sim \text{Po}(1)$ be independent random variables, all with the unit-mean Poisson distribution.

a) (8) Find the logarithm of the ch.f. of $X_j$, $\phi(\omega) := \mathbb{E}[e^{i\omega X_j}]$:
$\psi(\omega) = \log \phi(\omega) =$

b) (6) For numbers $a > 0$, find the log ch.f. $\psi_1(\omega)$ of $(X_j - 1)/a$.
$\psi_1(\omega) =$

c) (6) Let $S_n = X_1 + \cdots + X_n$ be the partial sum. Find a sequence $a_n > 0$ such that the log characteristic function $\psi_n(\omega)$ of $(S_n - n)/a_n$ converges to $-\omega^2/2$ for every $\omega$, and explain what this says about the limiting probability distribution of $S_n$ (i.e., about the Po($n$) distribution for large $n$).\footnote{Recall the Taylor series $e^x = 1 + x + x^2/2 + o(x^2) \approx 1 + x + x^2/2$ near $x \approx 0$.}
Problem 6:Miscellaneous examples & counter-examples. Let \( \{X_n\} \), \( X \), and \( Y \) be real-valued RVs on a space \((\Omega, \mathcal{F}, P)\), and let \( \mu(dx) \) and \( \nu(dy) \) be the probability distribution measures of \( X \) and \( Y \), respectively.

a) (5) Suppose \( X \) and \( Y \) are independent, and \( g : \mathbb{R}^2 \to \mathbb{R} \) is a Borel function. Sometimes it’s okay to switch orders of integration to evaluate the expectation \( \mathbb{E}[g(X, Y)] = \iint g(x, y) (\mu \otimes \nu)(dx \, dy) \) as either of:

\[
\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(x, y) \mu(dx) \right\} \nu(dy) \quad \text{or} \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(x, y) \nu(dy) \right\} \mu(dx)
\]

and sometimes it’s not. What are the two different sets of broadly-applicable conditions on \( g, \mu, \nu \) given by Fubini’s Theorem, either of which will ensure equality of these two expressions?

1.

2.

b) (5) Even if \( \{X_n\} \) and \( X \) are in \( L_1 \), and \( X_n \to X \) in probability, it’s possible that \( \mathbb{E}X_n \) does not converge to \( \mathbb{E}X \) and that \( \mathbb{E}|X_n - X| \) does not converge to zero. Give an example of \( \{X_n\} \) and \( X \) in \( L_1 \) where \( X_n \to X \) (pr.) but \( L_1 \) convergence fails.
Problem 6 (cont’d): More miscellaneous examples & counter-examples.

c) (5) Give an example of an RV $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and Lebesgue $\mathbb{P}$ that is in $L_1$ but not in $L_2$.

$d(\omega) =$

d) (5) Give an example of a Martingale $(X_n, \mathcal{F}_n)$ with filtration $\mathcal{F}_n = \sigma\{X_j : 0 \leq j \leq n\}$ and a (finite) stopping time $\tau$ for which $\mathbb{E}[X_0] \neq \mathbb{E}[X_\tau]$. 
Problem 7: More miscellany.

a) (10) The standard Cauchy distribution $\text{Ca}(0, 1)$ has pdf

$$f(x) = \frac{1/\pi}{1 + x^2}, \quad x \in \mathbb{R}$$

and famously has no mean, with $E[|X|] = \infty$ for $X \sim \text{Ca}(0, 1)$. For any $0 \leq p < 1$, however, $\|X\|^p = E[|X|^p] < \infty$. Find and prove\(^2\) a (numerical) finite upper bound

$$E[|X|^{1/2}] \leq \phantom{\text{_________________}}$$

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\(^2\) Suggestion: First use symmetry to focus on $\mathbb{R}_+$; then worry separately about $[0, 1]$ and $(1, \infty)$. 

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Problem 7 (cont’d): Yet more miscellany. Will it never end?

b) (5) Let \(X, Y\) be RVs on \((\Omega, \mathcal{F}, \mathbb{P})\), with \(X \in L_4\) and \(Y \in L_p\). For which \(p > 0\) is \(XY \in L_1\)? Why?

c) (5) If sequences \(\{X_n\}\) and \(\{Y_n\}\) of RVs on \((\Omega, \mathcal{F}, \mathbb{P})\) satisfy
\[
\mathbb{P}[X_n > Y_n] \leq 2^{-n}
\]
for each \(n \in \mathbb{N}\), does it follow that \(\limsup X_n \leq \liminf Y_n\) almost surely? Give a proof or counter-example.
Problem 8: Circle True or False; no explanations are needed.

a) T F If $X_n \to X$ (pr.) then $\limsup_{n \to \infty} X_n = X$.

b) T F If $X$ on $(\Omega, \mathcal{F}, P)$ has a cont. dist'n then $\Omega$ is uncountable.

c) T F If $g(\cdot)$ is a bounded Borel function on $\mathbb{R}$ and $X_n \to X$ (pr.) then $g(X_n) \to g(X)$ (pr.).

d) T F If $0 < X < \infty$ and $E[1/X] = 1/E[X]$ then $X \in L_\infty$.

e) T F If $X \perp Y$ and $P[X < Y] = P[X > Y] = 1/2$ then $X, Y$ have the same distribution.

f) T F If $X \perp Z$ and $Y \perp Z$ then $(X + Y) \perp Z$.

g) T F If $X \perp Z$ and $Y = e^X$ then $Y \perp Z$.

h) T F If probability measures $P, Q$ agree on a $\lambda$-system $\mathcal{L}$ then they agree on the $\pi$-system $\mathcal{P} = \pi(\mathcal{L})$ it generates.

i) T F If $X^* := \limsup_{n \to \infty} X_n$ is a non-constant random variable, then $\{X_n\}$ cannot be independent.

j) T F If $X$ has a discrete dist’n and $Y$ has a continuous one, then $(X + Y)$ must have a continuous distribution (even if $X, Y$ are not independent).
Blank Worksheet
Another Blank Worksheet
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
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</thead>
<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha^3}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
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<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\alpha}{\lambda}$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p q^{x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
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<tr>
<td></td>
<td></td>
<td>$f(y) = p q^{y-1}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(x) = \binom{n}{x} \left(\frac{A}{A+B}\right)^x \left(\frac{B}{A+B}\right)^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$nP$</td>
<td>$nP(1-P)\frac{(n-x)(A+B)}{n}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta \sqrt{2\pi \sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2/3$</td>
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<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-[\log(x-\mu)]^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2}(e^{-\sigma^2}-1)$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{\alpha-1}{x} p^\alpha q^{\alpha-x} \cdot q^{-1} \frac{1}{\Gamma(\alpha)}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
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<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = (\alpha/\epsilon)(1 + x/\epsilon)^{-\alpha-1}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\epsilon}{\alpha-1}$</td>
<td>$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$</td>
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<td>$f(y) = \alpha y^{\alpha-1} g^{\alpha+1}$</td>
<td>$y \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon \alpha}{\alpha-1}$</td>
<td>$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$</td>
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<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
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<tr>
<td>Snedecor $F$</td>
<td>$\text{F}(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)} \cdot \frac{1}{\nu_2^2} x^\frac{\nu_2}{2} \left(1 + \frac{\nu_2}{\nu_1} x\right)^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_1-2)}$</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi \nu}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu}{2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0^*$</td>
<td>$\nu/(\nu-2)^*$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$</td>
<td>$\frac{\Gamma(1+2/\alpha)}{\beta^{2/\alpha}} \frac{1}{\beta^{1/\alpha}}$</td>
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</tbody>
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