Conditional Expectation

1. Let \( \{N_t\}_{t \geq 0} \) be a homogeneous Poisson process with rate \( \lambda \), so \( N_0 = 0 \) and for every \( n \in \mathbb{N} \) and \( 0 = t_0 < t_1 < ... < t_n < \infty \) the random variables \( X_i := [N_{t_i} - N_{t_{i-1}}] \) for \( 1 \leq i \leq n \) are independent with marginal distributions \( X_i \sim \text{Po}(\lambda(t_i - t_{i-1})) \). For \( 0 < s < t < \infty \) find the conditional expectations:

\[
E[N_s \mid N_t] = \quad E[N_t \mid N_s] =
\]

2. Let \( \{X_1, X_2\} \text{id} \sim \text{Ex}(1) \) be iid unit-rate exponential random variables, \( t > 0 \). Find:

(a) \( E[X_1 \mid X_1 + X_2] = \)
(b) \( P[X_1 < 3 \mid X_1 + X_2] = \)
(c) \( E[X_1 \mid X_1 \wedge t] = \)
(d) \( E[X_1 \mid X_1 \vee t] = \)

3. Let \( X, Y \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) and suppose \( E[X \mid Y] = \phi(Y) \) for a monotonically decreasing Borel function \( \phi : \mathbb{R} \to \mathbb{R} \). Prove that \( \text{Cov}(X, Y) \leq 0 \).

4. If \( X \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) and \( \mathcal{H} \subset \mathcal{G} \subset \mathcal{F} \), show

\[
E\left[ (X - E[X \mid \mathcal{H}])^2 \right] \geq E\left[ (X - E[X \mid \mathcal{G}])^2 \right]
\]

In English, the MSE of the conditional expectation \( E[X \mid \mathcal{G}] \) given the bigger \( \sigma \)-algebra \( \mathcal{G} \) is smaller than the MSE of the conditional expectation \( E[X \mid \mathcal{H}] \) given the smaller \( \sigma \)-algebra \( \mathcal{H} \).

What does this imply for the trivial \( \sigma \)-algebra \( \mathcal{H} = \{\emptyset, \Omega\} \)?