Martingales

A sequence \( \{(X_n, \mathcal{F}_n), \ n \geq 0 \} \) of random variables \( X_n \in L_1(\Omega, \mathcal{F}, P) \) and a nested sequence of \( \sigma \)-algebras \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F} \) is a martingale if for each \( 0 \leq n \leq m < \infty \)

\[
X_n = \mathbb{E}[X_m \mid \mathcal{F}_n],
\]
so on average the sequence neither increases nor decreases. Note this implies that \( X_n \) is \( \mathcal{F}_n \)-measurable. By the tower property of conditional expectations, it is enough to check this condition for \( m = n + 1 \).

1. A sequence \( \{(X_n, \mathcal{F}_n), \ n \geq 0 \} \) is predictable if \( X_{n+1} \) is \( \mathcal{F}_n \)-measurable for each \( n \). Show every predictable martingale is constant (i.e., \( X_n = X_0 \) a.s.).

2. A sequence \( \{(X_n, \mathcal{F}_n), \ n \geq 0 \} \subset L_1(\Omega, \mathcal{F}, P) \) is a submartingale if, for each \( 0 \leq n \leq m < \infty \), \( \mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F} \) and \( X_n \leq \mathbb{E}[X_m \mid \mathcal{F}_n] \)— so, on average, \( X_n \) is increasing. Let \( \{(X_n, \mathcal{F}_n), \ n \geq 0 \} \) and \( \{(Y_n, \mathcal{F}_n), \ n \geq 0 \} \) be submartingales on \( (\Omega, \mathcal{F}, P) \). Show that their max \( (X_n \vee Y_n) \) and sum \( (X_n + Y_n) \) are submartingales too.

3. Show that every submartingale \( \{(X_n, \mathcal{F}_n) \} \) can be written as the sum \( X_n = (M_n + A_n) \) of a martingale \( \{(M_n, \mathcal{F}_n) \} \) and a predictable non-decreasing process \( A_n \), and that the decomposition is unique if we set \( A_0 = 0 \). Suggestion: How must \( A_n \) be defined to make \( A_0 = 0 \) and \( \mathbb{E}[(X_{n+1} - A_{n+1}) \mid \mathcal{F}_n] = (X_n - A_n) \)?

4. Let \( \{(M_n, \mathcal{F}_n) \} \) be a martingale and \( \phi : \mathbb{R} \to \mathbb{R} \) a convex function for which \( X_n := \phi(M_n) \) is in \( L_1(\Omega, \mathcal{F}, P) \). Show that \( \{(X_n, \mathcal{F}_n) \} \) is a submartingale.

5. Fix \( 0 < p < 1 \), set \( q := 1 - p \), and let \( \{\xi_j\} \) be iid random variables with \( P[\xi_j = 1] = p \) and \( P[\xi_j = -1] = q \). Set \( S_n := \sum_{j \leq n} \xi_j \), a random walk (possibly an asymmetric one) on the integers starting at \( S_0 = 0 \).

(a) For which \( \alpha \in \mathbb{R} \) is \( X_n := [S_n - \alpha n] \) a martingale?
(b) For which \( \alpha, \beta \in \mathbb{R} \) is \( Y_n := [(S_n - \alpha n)^2 - \beta n] \) a martingale?
(c) For which \( r > 0 \) is \( Z_n := [r^{S_n}] \) a martingale?
(d) Is \( S_n \) a submartingale?

Of course the answers will depend on \( p \).
Extremes

Most computations about extremes depend in one way or another on the elementary limit \((1 + z/n)^n \to e^z\) — the trick is arranging for \(\mathbb{P}[X_n^* \leq x]\) to look like \((1 + z/n)^n\).

6. Let \(\{X_j\}_{j \in \mathbb{N}} \sim \text{Ex}(\lambda)\) be independent exponentially-distributed random variables, and let

\[X_n^* := \max\{X_j : 1 \leq j \leq n\}\]

be the maximum of the first \(n\). Find sequences \(\{a_n, b_n\}\) of real numbers such that

\[Y_n := \frac{X_n^* - b_n}{a_n} \Rightarrow \text{Gu}(0, 1),\]

the standard Gumbel distribution. The \(\text{Gu}(m, s)\) distribution has CDF

\[F(x) = \exp\left(-e^{-(x-m)/s}\right)\]

for \(x \in \mathbb{R}\). The maxima from many other distributions (normal, gamma, etc.) are also approximately Gumbel (you don’t have to prove that!).

7. The standard Student’s \(t\) distribution with \(\nu\) degrees of freedom, \(t_\nu\), has pdf

\[f_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} (1 + t^2/\nu)^{-(\nu+1)/2}\]

for \(t \in \mathbb{R}\). Let \(\{T_j\} \sim t_\nu\) and set \(T_n^* := \max\{T_j : 1 \leq j \leq n\}\), and find sequences \(\{a_n, b_n\}\) of real numbers such that \((T_n^* - b_n)/a_n\) converges in distribution to some limit. What’s the limiting distribution? You don’t need to prove it, but this is also the limiting distribution for the maxima of many other heavy-tailed distributions (Pareto, \(\alpha\)-Stable, log Normal, etc.).

Hints: (1) Don’t get mesmerized by the normalizing constant, and (2) if \(x\) is huge and \(p\) is any fixed power then \((x+1)^{-p}\) and \((x)^{-p}\) don’t differ by much.