Yesterday in the Review Session I punted on answering Problem 7 on the 2016 Final Exam. Simplifying a little (by conditioning on the event $Z \neq 0$), the problem is:

Let $X, Y \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$ and set $Z := 3X + 4Y$. Find $E[X \mid Z]$.

The joint distribution of $X$ and $Z$ is bivariate Gaussian, so (as the footnote hint suggests) the solution must be of the form

$$E[X \mid Z] = a + bZ$$

for some $a, b \in \mathbb{R}$. We can find $a, b$ by projection or, equivalently, by using the fact that

$$E\left[ E[X \mid Z] \cdot g(Z) \right] = E\left[ X \cdot g(Z) \right]$$

for any Borel indicator function $g(\cdot)$ (that's the definition of conditional expectation) or, by linearity and LDCT, any Borel $g(\cdot)$ for which $X \cdot g(Z) \in L_1$ — in particular, for $g(z) := 1$ and $g(z) := z$:

$$E[(a + bZ) \cdot 1] = E[X \cdot 1]$$

$$= 0,$$ so

$$a = 0.$$  

$$E[(a + bZ) \cdot Z] = E[X \cdot Z]$$

$$= E[X(3X + 4Y)]$$

$$= 3,$$ so

$$b \ E[Z^2] = 3$$

$$b = 3/25$$

since $E[X^2] = 1$, $E[XY] = 0$, and $E[Z^2] = 3^2 + 4^2 = 25$. Thus

$$E[X \mid Z] = (3Z/25) = 0.12Z.$$

If we had $Z = X_1 + X_2 + X_3 + Y_1 + Y_2 + Y_3 + Y_4$ with all the $\{X_i, Y_j\} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$, then each $E[X_i \mid Z]$ and $E[Y_j \mid Z]$ would be $Z/7$ by symmetry, making it tempting to guess in Problem 7 that $E[X \mid Z]$ is $Z/7$ (or maybe $3Z/7$ or $4Z/7$), but we have to resist some temptations, the answer is $(3/25)Z$. 
2c) Let $X_n \to X$ pr. and $Y \in L_2$. If $X_n \leq Y$, is $E[X] \geq \limsup E[X_n]$?

If we had $X_n \to X$ almost surely, then we would have $X = \limsup X_n$ and $Y \in L_2 \subseteq L_1$ so the result would be “yes” by Corollary 2 to Fatou’s Lemma in the Week 4 class notes: Since $X_n \leq Y \in L_1$, Fatou’s Lemma says the nonnegative RVs $Y_n := [Y - X_n] \geq 0$ satisfy

$$\liminf E[Y_n] \geq E[\liminf Y_n], \quad i.e.,$$

$$E[Y] - \limsup E[X_n] \geq E[Y] - E[\limsup X_n],$$

so

$$E[X] = E[\limsup X_n] \geq \limsup E[X_n].$$

BUT, because we only have convergence pr., the relation “$X \nRightarrow \limsup X_n$” may fail and we need to do a little more work.

Suppose, for contradiction, that $E[X] < \limsup E[X_n]$. Then for some $\epsilon > 0$ and some subsequence $n_i \to \infty$,

$$E[X] + \epsilon \leq E[X_{n_i}]$$

for each $i \in \mathbb{N}$. Now, since $X_{n_i} \to X$ (pr.), find a further subsequence $n_{ij}$ along which $X_{n_{ij}} \to X$ (a.s.). But $X_{n_{ij}} \leq Y \in L_1$ and $X = \limsup X_{n_{ij}}$ almost surely, so Corollary 2 to Fatou’s lemma says

$$E[X] = E[\limsup X_{n_{ij}}] \geq \limsup E[X_{n_{ij}}] \geq E[X] + \epsilon,$$

a contradiction. Thanks to Shuxi Zeng for spotting a gap in an earlier version of this note.

-RLW