These problems are typical of the kind of calculations we will encounter throughut the course. You should see how many of them you can do without help. Your calculus text will contain the material needed to do them; solutions are given on the back of this page.

1. Find $\int_0^A e^{-cx}\, dx$ in terms of A and c, assuming A is positive.
2. Why is $\int_0^\infty e^{-cx}\, dx$ called an *improper* integral? How is its value defined? If c is positive, what is its value? What if c is zero or negative?
3. Find (a) $\int_0^A x\, e^{-cx}\, dx$ and (b) $\int_0^\infty x\, e^{-cx}\, dx$ for all $A > 0$ and $-\infty < c < \infty$.
4. Find $\int_0^\infty x\, e^{-x^2}\, dx$.
5. What is meant by the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$? What is the sum (in terms of x)? How would you show that your answer is correct?
6. Find the sum of the series $(4/5)^3 + (4/5)^4 + (4/5)^5 + \cdots + (4/5)^{20}$.
7. What is the sum of the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$? For which x does it converge?
8. Give (a) the binomial expansion of $(a + b)^n$ and (b) the coefficient of x^3 in $(x + 1)^{52}$.
9. For a positive integer n, what is the value of $1 + 2 + 3 + \cdots + n$?
10. Find (a) $\lim_{x\to\infty} \frac{x}{e^x}$, (b) $\lim_{x\to\infty} \frac{x^2}{e^x}$ and (c) $\lim_{x\to\infty} \frac{x^k}{e^x}$ for an arbitrary positive integer k.
11. Find (a) $\lim_{n\to\infty}(1 - 2/n)^n$ and (b) $\lim_{n\to\infty}(a + b\,n)/(c + d\,n)$.
12. Sketch the triangle in the plain whose vertices are at $(0, 0)$, $(0, 1)$ and $(1, 0)$. Find the integral of $f(x, y) = x + xy$ over that triangle.
13. Find the integral of $f(x, y) = e^{-(x^2+y^2)}$ over the first quadrant in the plane $\mathbb{R}^2_+ = \{(x, y) :\ x \geq 0,\ y \geq 0\}$ (Hint: Use polar coordinates).
14. For which $c \in \mathbb{R}$ does the integral $\int_0^\infty \frac{\arctan x}{x^c}\, dx$ converge?

Solutions

1. The integral equals $-\frac{1}{c}e^{-cx}\big|_0^A = \frac{1}{c}(1-e^{-cA})$, but this is only valid if $c \neq 0$. If $c = 0$ the integral is $\int_0^A 1\,dx = A$.

2. It is *improper* because one of the limits of integration is infinite or, more precisely, because the interval of integration is unbounded. It would also be improper if the *integrand* were unbounded, like $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$. Its value is defined to be $\lim_{A\to\infty}\int_0^A e^{-cx}\,dx$, which is $1/c$ for $c > 0$ and ∞ for $c \le 0$.

3. Integrate twice by parts for $c \neq 0$ to find (a) $\int_0^A x\,e^{-cx}\,dx = c^{-2}\big[1-(1+c\,A)e^{-cA}\big]$, while for $c = 0$, $\int_0^A x\,e^{-cx}\,dx = A^2/2$. For $c > 0$, (b) $\int_0^\infty x\,e^{-cx}\,dx = c^{-2}$, or ∞ for $c \le 0$.

4. Notice that $(e^{-x^2})' = -2xe^{-x^2}$. If you multiply both sides by $(-1/2)$ you find that $\int_0^\infty x\,e^{-x^2}\,dx = (-1/2)e^{-x^2}\big|_0^\infty = 1/2$.

5. Call the partial sum of this Geometric Series $S_n = 1 + x + x^2 + x^3 + \cdots + x^n$; the infinite series is defined to be the limit $S = \lim_{n\to\infty} S_n$. It's easy to show that the finite sum is $S_n = (1-x^{n+1})/(1-x)$, if $x \neq 1$, or $S_n = n+1$, if $x = 1$ (verify by multiplying both sides by $(1-x)$ and cancelling the "telescoping series"), so the infinite sum is $S = \lim_{n\to\infty}(1-x^{n+1})/(1-x) = 1/(1-x)$ if $|x| < 1$, while the limit is infinite for $x \ge 1$ and undefined for $x \le -1$. The way to remember it is: *First term included, minus first term omitted, divided by one minus the ratio.* This is a good one to memorize.

6. By the recipe above, it's $\left[(4/5)^3 - (4/5)^{21}\right]/(1-4/5) = 2.56 - 0.04611686 = 2.513883$.

7. This is the exponential series, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$. It converges for all real (or even complex) numbers.

8. The binomial theorem gives $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$. With $n = 52$, $a = x$, and $b = 1$ we find that the coefficient of x^3 in $(x+1)^{52}$ is $\binom{52}{3} = (52\times 51\times 50)/(3\times 2\times 1) = 22100$.

9. For positive integers n, $S_n := 1 + 2 + 3 + \cdots + n = n(n+1)/2$. You can verify this by induction, or can guess it by looking for any quadratic function $\alpha n^2 + \beta n + \gamma$ that works for $n = 0, 1, 2$, but a great way to see and remember it is to write the sequence $1, 2, 3, \cdots, n$ above the sequence $n, (n-1), \cdots, 3, 2, 1$ and see that $2S_n$ is the sum of n terms, each $n+1$.

10. $\lim_{x\to\infty} x/e^x = \lim_{x\to\infty} x^2/e^x = \lim_{x\to\infty} x^k/e^x = 0$ for any real number k. You can show this by l'Hôspital's rule, or can just remember that exponentials always beat polynomials.

11. $\lim_{n\to\infty}(1 + x/n)^n = e^x$ for any (real or complex) x. In (a) we have $x = -2$, so the limit is e^{-2}. For (b), either l'Hôspital's rule or dividing numerator and denominator by n will lead to the same limit, b/d, if $d \neq 0$. Limit is ∞ if $d = 0$ and $b/c > 0$; $-\infty$ if $d = 0$ and $b/c < 0$; a/c if $b = d = 0$ and $c \neq 0$; and undefined if $c = d = 0$.

12. The double integral can be set up in either of two ways, integrating first with respect to x or to y. If we choose to integrate over horizontal lines (first dx, then dy) we find that for

each $0 \le y \le 1$ the range of x in the triangle is $0 \le x \le 1-y$, so

$$\iint_\Delta f(x,y)dx\,dy = \int_0^1 \left[\int_0^{1-y} (x+xy)\,dx\right] dy = \int_0^1 \left[(1+y)(1-y)^2/2\right] dy = 5/24.$$

If instead we choose to integrate over vertical lines (first dy, then dx) we find that for each $0 \le x \le 1$ the range of y in the triangle is $0 \le y \le 1-x$, so

$$\iint_\Delta f(x,y)dy\,dx = \int_0^1 \left[\int_0^{1-x} (x+xy)\,dy\right] dx = \int_0^1 \left[x(1-x)+x(1-x)^2/2\right] dx = 5/24.$$

13. Switch to polar coordinates and apply the answer from 4. above:

$$\iint_{\mathbb{R}^2_+} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta = (\pi/2)(1/2) = \pi/4.$$

14. Since the integrand is continuous, the only possible trouble-spots are near zero and near infinity. Near zero $\arctan x \approx x$ so the integrand is $\approx x^{1-c}$ and is integrable only for $c < 2$. Near infinity, $\arctan x \approx \pi/2$ so the integrand is $\propto x^{-c}$ and is integrable only for $c > 1$. Thus, the integral converges for $1 < c < 2$.