Midterm Examination #1

Mth 135 = Sta 104^c

Tuesday, October 19, 2010
2:50 – 4:05 pm

- ◦ This is a **closed book** exam— put your books on the floor.
- ◦ You may use a calculator and **one page** of your own notes.
- ◦ Do not share calculators or notes. **No cell phones**.
- ◦ **Show** your **work**. Neatness counts. $\boxed{\text{Boxing}}$ answers helps.
- ◦ Numerical answers: **four significant digits** or fractions **in lowest terms**. Nonnumerical: **Simplify!**
- ◦ PDF and normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ____________________

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Joey the Kangaroo wins a fraction p of his boxing matches against tourists. Let X be his number of wins in n independent trials, so X takes on the each integer in the range $x \in \{0, 1, ..., n\}$ with probability

$$f(x) = \Pr[X = x] = \binom{n}{x} p^x \, q^{n-x}, \qquad x = 0, 1, 2, ..., n$$

where $q = 1 - p$. For each n find:

a) (10) What is the largest value of p such that zero is the most probable outcome for X?

$(\forall x) \quad \mathsf{P}[X = 0] \geq \mathsf{P}[X = x]$ for $0 \leq p \leq$ ____________

b) (10) For each $0 \leq p \leq 1$ find the expected value of $g(X)$ for the function $g(x) = x!\,(n - x)!$:

$\mathsf{E}[X!\,(n - X)!] =$ ____________

Problem 2: Dave the Demented Tasmanian Devil rolls a fair Die repeatedly until an ace (*i.e.*, one) appears; the random variable D is the **number of non-aces** that occur before that first ace. Meanwhile Carl the Crafty Echidna tosses a fair Coin until a Head appears; the random variable C is the **number of tails** that occur before that first head.[1] Find:

a) (4) The probability that D is no larger than C:
$\mathsf{P}[D \le C] =$

b) (12) The following expectations and variances[2]

$\mathsf{E}[D + C] =$ ______________ $\mathsf{E}[2 - 10C] =$ __________________

$\mathsf{E}[D \times C] =$ __________________ $\mathsf{Var}[D - 2C] =$ __________________

c) (4) Now find the *conditional* probability distribution for D, given that $D \le C$. For each $d \in \mathbb{R}$, give:
$\Pr[D = d \mid D \le C] =$

[1]Mnemonic: **D** = **D**ie, **C** = **C**oin. Like kangaroos, Tasmanian devils and echidnas are Australian mammals. So are bandicoots, dunnarts, koalas, platypi, quolls, wallabys, wombats, and hundreds of others.

[2]The means, variances, and pdfs or pmfs for common distributions are included on a sheet at the back of this exam.

Problem 3: Cam practices free-throws by shooting until he has made 100 baskets (then he gets to eat dinner). If his shots are independent, and he makes 80% of his tries, the total number X of *misses* can be viewed as a sum $X = \sum_{i=1}^{100} X_i$ of the number of misses X_1 before the first basket, plus the additional number of misses X_2 before the second basket, and so on.

a) (8) What is the *exact* distribution for the number of **misses** before Cam eats dinner? Give the name of the distribution and its mean and variance[3]

Dist'n Name: ______________ $\mu =$ ________ $\sigma^2 =$ ________

b) (8) By the Central Limit Theorem, X has *approximately* a normal distribution. Find the approximate probability that Cam misses at least 35 baskets before completing his practice

$\Pr[X \geq 35] \approx$

c) (4) Cam's brother Tyler also makes 80% of his free-throws, but Ty shoots until he misses two shots *in a row*. On average, who takes more shots— Cam or Tyler? Find the expected number of *shots* for each.[4]

E[Cam's shots] = ____________ E[Tyler's shots] = ____________

[3]Note that a page of pdfs or pmfs, means, and variances for common distributions is attached to this exam.

[4]Finding Tyler's shots is the hardest question on the exam, and it only counts 3 pts.

Problem 4: A bag contains twenty-five Scrabble tiles with 25 different letters (we lost the "Y" tile)— five vowels (A, E, I, O, U) and 20 consonants.

a) (5) What is the probability that five draws *with replacement* will all show different tiles?

Pr[All five different] =

b) (5) What is the probability that five draws *with replacement* will include no vowel?

Pr[No vowel in 5 draws, with repl.] =

c) (5) What is the probability that five draws *without replacement* will include no vowel?

Pr[No vowel in 5 draws, no repl.] =

d) (5) What is the expected value of X, the number of draws *without replacement* required to draw all five vowels??[5].

$\mathsf{E}[X] =$

[5]Hint: Indicator variables might help— let I_i be **one** if the ith tile is drawn at or before the draw of the last vowel, and **zero** otherwise.

Problem 5: Two points each, no explanations needed:

a) If X, Y indep., then $\mathsf{Var}[X+2Y] = \mathsf{Var}[X] + 2\mathsf{Var}[Y]$ T F

b) If $\mathsf{E}[X^2] = 4$ then $\mathsf{E}[|X|] \le 2$. T F

c) If X and Y are independent, then $\mathsf{E}[e^{X-Y}] = \mathsf{E}[e^X]\,\mathsf{E}[e^{-Y}]$ T F

d) If X is Poisson with $\mathsf{P}[X=0] = \frac{1}{2}$, find $\mathsf{P}[X=1] =$ ________

e) If $\mathsf{E}[X \cdot Y] = \mathsf{E}[X] \cdot \mathsf{E}[Y]$, then X, Y are independent T F

f) Find & simplify: $\sum_{k=3}^{9} 7 \cdot 2^k =$ ________

g) If $\mathsf{P}[X=k] = \frac{1}{3}$ for $k \in \{-1, 0, 1\}$ find: $\mathsf{E}[2^X] =$ ________

h) If an event A occurred, then $\mathsf{P}[A] > 0$ T F

i) If $\mathsf{P}[X > Y] = 0$ then $\mathsf{E}[X] \ge \mathsf{E}[Y]$ T F

j) If $\mathsf{P}[A] \ge \mathsf{P}[B] > 0$ then $\mathsf{P}[A \mid B] \ge \mathsf{P}[A]$ T F

Extra worksheet, if needed:

Another extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

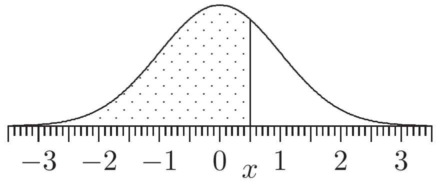

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,G,B)$	$f(x)=\frac{\binom{G}{x}\binom{B}{n-x}}{\binom{G+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{G}{G+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\big(e^{\sigma^2}-1\big)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\mu)$	$f(x)=\frac{\mu^x}{x!}e^{-\mu}$	$x\in\mathbb{Z}_+$	μ	μ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	