Midterm Examination # 2

Mth 135 = Sta 104

Thursday, November 18, 2010
2:50 – 4:05 pm
Version *a*

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes.
- **Show** your **work**. Neatness counts. $\boxed{\text{Boxing}}$ answers helps.
- Numerical answers: **four significant digits** or fractions **in lowest terms**. Simplify *all* answers.
- Extra worksheet and pdf & normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ____________________

Print **Name**: ____________________

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2.	/20
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4.	/20
5.	/20
Total:	/100

Problem 1: The independent random variables X and Y have uniform distributions on the interval $[0, 1]$. Pictures help (I'll draw one of them for you). Find:

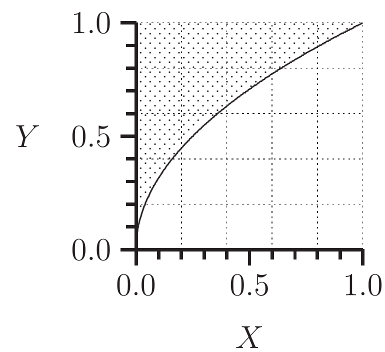
a. $\mathsf{P}[X > 2Y] \quad =$

b. $\mathsf{P}[X > Y^2] \quad =$

c. $\mathsf{E}[e^{Y-X}] \quad =$

d. $\mathsf{E}[\sqrt{X/Y}] \quad =$

e. $\mathsf{E}[Y \mid X \le Y^2] \quad =$



Hint: *No* 2-dimensional integration is needed, just a little plane geometry.

Problem 2: Let X_t be the number of fish caught in the first t hours if we catch $\lambda = 4$ fish per hour, on average[1], with the usual assumptions[2]. In each case below pick the best possible distribution from those suggested (Beta, Binomial, Exponential, Gamma, Geometric, Hypergeometric, Negative Binomial, Normal, Poisson, Uniform) and give its mean and variance (remember λ was given— **give** correct **units**, and simplify!):

a. The time T_1 until the first fish is caught:
○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
$\mu =$ $\sigma^2 =$

b. Of the first 10 fish, the number that took more than 15 minutes (each) to catch:
○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
$\mu =$ $\sigma^2 =$

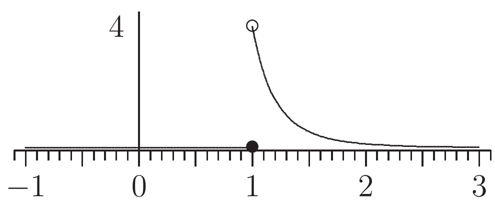
c. The time T_6 needed to catch six fish:
○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
$\mu =$ $\sigma^2 =$

d. The number of fish caught in less than 15 minutes each, BEFORE the first that took more than 15 minutes to catch:
○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
$\mu =$ $\sigma^2 =$

e. The total weight of the first 100 fish caught, if fish weigh 8oz on average with a standard deviation of 2oz?
○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
$\mu =$ $\sigma^2 =$

[1] or: The number of potholds in t km of road; or the number of falling stars in t square degrees of the sky in a Perseid meteor shower; or the number of rain drops hitting your nose in t minutes; or ...

[2] independent event counts in disjoint intervals; expected count depends only on length of interval



Problem 3: The random variables X has pdf:

$$f_X(x) = \begin{cases} 6/x^c & x > 1 \\ 0 & x \le 1. \end{cases}$$

a) Find the value of c. Show your work.
$c =$

b) Find the pdf for $Y = X^2$, correctly for all $y \in \mathbb{R}$:
$f_Y(y) =$

Problem 3 (cont'd): Still X has pdf $f_X(x) = 6/x^c$ for $x > 1$; by now you know c:

c) Find the hazard function $h(x)$, so $\mathsf{P}[X \le x + \epsilon \mid X > x] \approx \epsilon h(x)$ for each $x \in \mathbb{R}$ and small $\epsilon > 0$:

$h(x) =$

d) Suppose X is the life-time (in years) of some replacable item, like a light-bulb. Amazingly the probability of failure is *zero* during the first year... but which is more likely to fail in the next short interval (say, a day or two): ◯ a three-year old one, or ◯ a two-year-old one, both still working? By how much? (give approximate ratio of failure probabilities):

$$\lim_{\epsilon \to 0} \frac{\mathsf{P}[\text{ Fail in } \epsilon \text{ yrs } \mid \text{ Three Yrs Old }]}{\mathsf{P}[\text{ Fail in } \epsilon \text{ yrs } \mid \text{ Two Yrs Old }]} \approx$$

Problem 4: The random variable X has the standard exponential distribution, with probability density function and cumulative distribution function

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-x} & \text{for } x > 0 \end{cases} \qquad F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-x} & \text{for } x > 0 \end{cases}$$

Find the density functions for the following random variables at **all** $y \in \mathbb{R}$:

a. $Y_1 = e^{-X}$: $\quad f_1(y) =$

b. $Y_2 = 2X$: $\quad f_2(y) =$

c. $Y_3 = 1/X$: $\quad f_3(y) =$

d. $Y_4 = (X-2)^2$: $\quad f_4(y) =$

Problem 5: The random variables X and Y have a bivariate normal distribution, with means and variances given by:

$$\mathsf{E}[X] = 4 \quad \mathsf{Var}[X] = 9 \quad \mathsf{E}[Y] = 10 \quad \mathsf{Var}[Y] = 61 \quad \mathsf{Cov}[X, Y] = 15.$$

Without doing any integration, find each of the following:

a. (5) $\mathsf{E}[X^2 - Y^2] =$ ____________________

b. (5) $\mathsf{P}[5X < 13 + 2Y] =$ ____________________

c. (5) $\mathsf{Var}[2X - Y] =$ ____________________

d. (5) $\mathsf{E}[X \times Y] =$ ____________________

Extra worksheet, if needed:

Another extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

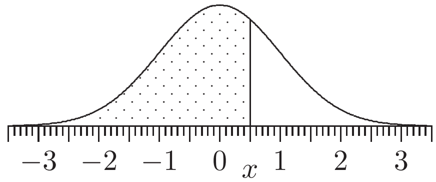


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$
$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	