Final Examination

Mth 230 = Sta 230 = Mth 730

Wednesday, December 12, 2018
9:00am – 12:00n

- This is a **closed book** exam— put your books and bags on the floor.
- You may use a calculator and **two pages** of your own notes.
- Do not share calculators or notes. **No phones** or other network-connected devices may be used.
- **Show** your **work**. Neatness counts. $\boxed{\text{Boxing}}$ answers helps.
- Numerical answers: **four sig**nificant **dig**its or fractions in lowest terms. For full credit you must **simplify** *all* expressions.
- Extra worksheet and pdf & normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ______________________________

Print Name Clearly: ____________________

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
	/100		/100
Total:	/200		

Problem 1: The random variables X, Y have joint pdf $f(x,y)$ on $\mathbb{R}^2$. Show what integrals (or other expressions) would be needed to find each of the following quantities, all in terms of the joint pdf $f(x,y)$. Be careful about the *limits* of integration.

a) (3) $\mathsf{P}[X < Y + 2] = \int \quad \int$

b) (3) $\mathsf{P}[X < 2 \mid Y > 3] = \Bigg\{$

c) (3) $f_Z(z) = \int$ for $Z := X+Y$

d) (3) $\mathsf{P}[X < Y^2] = \int \quad \int$

e) (3) $f(y \mid x) =$

f) (3) $\mathsf{P}[X - Y \le z] = \int \quad \int$

g) (2) $\mathsf{P}[|X| \le \exp(Y)] = \int \quad \int$

Problem 2: The random variables $\{U_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,6)$ are independent with uniform distributions on $[0,6]$, and $S := \sum_{j=1}^{100} U_j$ is their sum.

a) (4) Find[1] the mean and variance for S:

$\mathsf{E}[S] =$ ____________________ $\mathsf{V}[S] =$ ____________________

b) (4) What does the Central Limit Theorem suggest for the approximate value of:
$\mathsf{P}\big[|S-300| > 50\big] \approx$

c) (4) What bound does Chebychev's (or Markov's) Inequality give for:
$\mathsf{P}\big[|S-300| > 50\big] = \mathsf{P}\big[|S-300|^2 > 2500\big] \le$

d) (4) For constants $c \in \mathbb{R}$, what is:
$\mathsf{E}[(S-c)^2] =$

e) (4) Find[2] the mean μ and variance σ^2 for $U_* := \min_{1\le j\le 10} U_j$.

$\mu =$ ____________________ $\sigma^2 =$ ____________________

[1]Note: See page 16 for a table of common distributions with their pdf/pmf s, CDF s, means, and variances. Using that, no integrals are needed for this problem.

[2]Hint: $U_* = 6X$ for a RV X with one of the distributions on Page 16, so no integration is needed. What is the dist'n of $(U_j/6)$? Of $X_* := \min\{(U_1/6), \cdots, (U_{10}/6)\}$?

Problem 3: The random variables X and Y have joint probability density function

$$f(x,y) = \begin{cases} c\,x^2 & 0 < x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) (4) Find the normalizing constant:

$c =$ ____________________

b) (8) Find the marginal density functions (for *all* x, y in $\mathbb{R}$) *or* specify the distribution's name and parameter(s) from Page 16:

$X \sim \Bigg\{$ $\qquad\qquad\qquad$ $Y \sim \Bigg\{$

Problem 3 (cont'd): X and Y still have joint pdf $f(x,y) = c\,x^2\mathbf{1}_{\{0<x\le y\le 1\}}$.

c) (8) Find the pdf for $Z := X/Y$ *or* specify the distribution's name and parameter(s) from Page 16:

$$Z \sim \left\{ \vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}} \right.$$

Problem 4: The random variables X, Y, and Z are independent, with distributions and pdf/pmf s

$$X \sim \mathsf{Ge}(p) \qquad\qquad Y \sim \mathsf{Po}(\lambda) \qquad\qquad Z \sim \mathsf{Ex}(\mu)$$
$$f(x) = p\,q^x,\ x \in \mathbb{N}_0 \quad g(y) = \frac{\lambda^y}{y!} e^{-\lambda},\ y \in \mathbb{N}_0 \quad h(z) = \mu \exp(-\mu z),\ z > 0$$

where $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. Find (and **simplify**):

a) (5) $\mathsf{P}[X > Y] =$

b) (5) $\mathsf{P}[Z > X] =$

Problem 4 (cont'd): Again with $X \sim \mathsf{Ge}(p)$, $Y \sim \mathsf{Po}(\lambda)$ and $Z \sim \mathsf{Ex}(\mu)$ independent, find and **simplify**:

c) (5) $\mathsf{P}[X = 2Y] =$

d) (5) $\mathsf{P}[Y = 2Z] =$

Problem 5: Let $\xi_j \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ be independent standard Normal random variables and, for real numbers a, b, c, set

$$X = a\xi_1 \qquad Y = b\xi_1 + c\xi_2.$$

a) (6) Find a, b, c so that X and Y each have variance $\sigma^2 = 4$ and $\mathsf{Cov}(X, Y) = 2$.

$a =$ ____________ $b =$ ____________ $c =$ ____________

b) (4) For $a = 6$, $b = 3$, and $c = 4$, find:

$\mathsf{E}[Y \mid X] =$ ________________

c) (4) For $a = 6$, $b = 3$, and $c = 4$, find:

$\mathsf{E}[X \mid Y] =$ ________________

d) (6) For $a = 6$, $b = 3$, and $c = 4$, find

$\mathsf{P}[X < Y + 5] =$ ________________

Problem 6: When you push the button on the Statistical Science coffee maker labeled "8 oz", the actual volume of coffee dispensed (measured in fluid ounces) is a random quantity V with a normal probability distribution $V \sim \mathsf{No}(\mu, \sigma^2 = 0.16)$, with μ an adjustible feature of the machine.

a) (5) How should μ be set to ensure that exactly 2% of cups will overflow a cup that can hold exactly eight ounces?
$\mu =$

b) (5) If $n = 100$ such cups are filled in a day, with μ set as above, what is the approximate probability that one or fewer cups will overflow (assuming independence)?

c) (5) If exaxtly two of $n = 100$ such cups overflow, what is the probability that these were the first two cups poured? Assume independence again.

d) (5) What are the mean and variance for the total volume (in ounces) $T = \sum_{j=1}^{100} V_j$ of those 100 (independent) servings? Give units.

$E[T] =$ ____________________ $\mathsf{V}[T] =$ ____________________

Problem 7: Bozo the clown is host of a children's television show. Bozo is taking a refresher course at Clown College, and must take a multiple-choice test. Each question has 5 possible answers, one correct and four incorrect. Bozo knows the correct answer for about 80% of this sort of question. For each question he will answer correctly, if he knows the answer; and otherwise will guess, with each possible answer equally likely.

a) (5) What is the probability that Bozo will answer the first question correctly?

$\mathsf{P}[\text{Prob 1 Correct}] =$ ______________

b) (5) If Bozo *does* answer Problem 1 correctly, what is the probability that he actually knows the answer?

$\mathsf{P}[\text{Knows answer} \mid \text{Prob 1 Correct}] =$ ______________

c) (6) Suppose that there are 25 questions on the exam. Let N be the number that Bozo answers correctly. Find the mean and variance:

$\mathsf{E}[N] =$ ______________ $\mathsf{V}[N] =$ ______________

d) (4) What is the *exact* probability distribution of N? Give its name and the values of any parameter(s). Half-credit for an *approximate* distribution with parameter(s). Make any assumptions you feel are needed.
$N \sim$

Problem 8: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, or *Uniform* and, whatever the distribution, give its mean μ:

a) (4) The number of times Bozo's squirting flower works correctly before it fails, if failures are independent and have probability 0.01:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

b) (4) The number of shoes with holes in the soles, from a random draw of 6 shoes (without replacement) from Bozo's collection of 20 shoes, 8 of which have holes:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

c) (4) The number of children in Row 6 who mistake Bozo for Ronald McDonald, if 25% of children make that mistake and if Row 6 holds 12 children today?
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

d) (4) The number of times Bozo laughs in an hour, if he laughs about once every five minutes and if the numbers of laughs in different periods are independent?
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

e) (4) The total amount of weight Bozo gains in his 200 working days a year, if daily gains are independent and average 0.01 *lb* per working day?
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

Problem 9: After his performance Bozo passes a hat for tips. The hat garners \$34 in ten US bills: four one-dollar bills and six five-dollar bills. Bozo pulls three bills from the hat at random (without replacement). Let X denote the denomination of the smallest bill drawn and let Y denote the denomination of the largest bill drawn. For example, if the three bills were $1, 5, 1$, then $X = 1$ and $Y = 5$.

a) (5) Find the joint probability mass function for X and Y:

$$p(x, y) = \left\{ \right.$$

b) (5) Find the marginal probability mass functions:

$$p_X(x) = \left\{ \right. \qquad\qquad p_Y(y) = \left\{ \right.$$

c) (5) Find the conditional probability mass function:

$$p(y \mid X = 1) = \left\{ \right.$$

d) (5) Are X and Y independent? Explain.

Problem 10: True or false? Circle one; each answer is worth 2 points. No explanations are needed. "iid" means Independent & Identically Distributed.

a) T F If $\{X_j\}$ are iid with mean $\mu \neq 0$, then $\frac{n}{\sum_{j=1}^{n} X_j}$ converges to $1/\mu$.

b) T F If the pdf and CDF for a random variable $T > 0$ satisfy $f(t) = 4\big(1 - F(t)\big)$ then T has an exponential distribution.

c) T F If $A \cap B = \emptyset$ then A and B are independent.

d) T F If $X \sim \mathsf{No}(\mu = 2, \sigma^2 = 4)$ and $Y := 3X$ then $Y \sim \mathsf{No}(\mu = 6, \sigma^2 = 12)$.

e) T F The length of time until an event has the Poisson distribution.

f) T F $\mathsf{P}\big[|X| < Y\big] = \int_{-y}^{y} \Big[\int_{|x|}^{\infty} f(x,y)dy\Big]\,dx$.

g) T F If $U \sim \mathsf{Un}(0,1)$ then $X := \sqrt{U}$ has pdf $f(x) = \mathbf{1}_{\{0<x^2<1\}}$

h) T F If $\mathsf{E}[XY] = \mathsf{E}[X] \times \mathsf{E}[Y]$ then X and Y are independent

i) T F If the cond'l pdf of X given Y is a function $f(x \mid y) = g(x)$ of only x, then X and Y are independent.

j) T F If X has MGF $M(t)$ then $Y := 2X$ has MGF $M(2t)$.

Extra worksheet, if needed:

Another extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

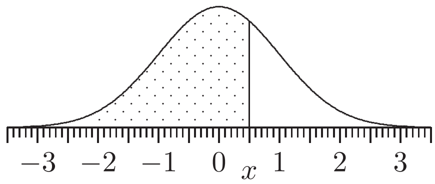

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	