

Midterm Examination # 1

STA 230 = MTH 230 = MTH 730: Probability

Wednesday, 2018 Oct 3
10:05 – 11:20am

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes. **No cell phones.**
- **Show your work.** Neatness counts. Boxing answers helps.
- For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify.**
- PDF and normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: _____

Print Name Clearly: _____

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Five indistinguishable coins are placed in a bag— three fair coins with $P[\text{Heads}] = 1/2$ and two two-headed coins with $P[\text{Heads}] = 1$.

a) (5) A single coin is drawn and tossed twice. Find the probability mass function (pmf) $f(x) := P[X = x]$ for the number X of Heads that appears.
 $f(x) =$

b) (5) A single coin is drawn and tossed twice. If $X = 2$ Heads appear, what is the probability the coin is fair?
 $P[\text{Fair} \mid X = 2] =$

c) (5) A coin is drawn from the bag, tossed, and replaced into the bag, twice. Now what is the pmf for the number X of Heads that appear?
 $f(x) =$

d) (5) A coin is drawn from the bag, tossed, and replaced into the bag, n times. Does the number X of Heads that appear have the **binomial distribution**? Why, or why not?

Problem 2: Let $X \sim \text{Po}(\lambda)$ have the Poisson probability distribution with mean parameter $\lambda > 0$.

a) (4) Recall that $e^z = 1 + z + z^2/2 + z^3/6 + \dots = \sum_{k=0}^{\infty} z^k/k!$, for any number z . Write out the first few terms and also give an infinite series for:
 $e^\lambda + e^{-\lambda} =$

b) (6) $P[X \text{ is even}] =$

c) (5) $E[2^X] =$

d) (5) $E[X!] =$

Problem 3: In a class there are 4 freshman boys, 6 freshman girls, 6 sophomore boys, and x sophomore girls for some number x . One student is selected at random (with all equally-likely) to present a homework solution at the board.

a) (4) For what value of x are the sex and class of the selected student independent?

b) (4) In how many ways can we order the ten freshmen, with the restriction that the girls precede the boys?

c) (4) For which values of x is $P[\text{Girl}] \geq P[\text{Freshman}]$?

d) (4) If $x = 14$ and four students are selected at random (equally-likely, *no* replacement), what is the probability that all are freshmen?

e) (4) If $x = 14$ and four times a student is selected at random and then replaced (equally-likely, *with* replacement), what is the probability that all four are girls?

Problem 4: A new vaccine is being compared to an older one using “matched pairs” as follows. Some even number $2n$ of subjects are arranged in n pairs who are similar in age, gender, disease state, *etc.* In each pair one subject is given the new vaccine and one the old one; neither the subjects nor the physician knows which is which. At the end of the trial one member of each pair is declared “better”, and we count the number X of pairs in which the new vaccine did better than the old one.

a) (6) If the new and old drugs are equally effective, does X have a **binomial** distribution? Why or why not?

b) (6) If X does have a $\text{Bi}(n, p)$ binomial distribution with $p = 1/2$, how large must k be for $P[X \geq k] \leq 0.01$ if $n = 400$? ¹

c) (8) If X does have a $\text{Bi}(n, p)$ binomial distribution with $p = 1/2$, how large must n be for the new vaccine to do better for at least 60% of the n subjects, with probability at most 1%, *i.e.*, for $P[X \geq 0.60 n] \leq 0.01$? ²

¹Note $\Phi(2.3263) = 0.99$.

²Here n will be big enough to skip the “continuity correction”, *i.e.*, the $\pm \frac{1}{2}$ terms.

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed. “iid” means Independent & Identically Distributed.

- a) T F If $P[A] > P[B] > P[AB] > 0$, then $P[A | B] > P[B | A]$.
- b) T F If $P[Y > X] = 1$, then $E[Y] > E[X]$.
- c) T F If $\{X, Y, Z\}$ are iid and $P[X < Y < Z] = 1/6$ then X has a continuous distribution.
- d) T F If $P[A] + P[B] > 1$, then $A \cap B \neq \emptyset$.
- e) T F If $X \sim \text{No}(0, 1)$ then $P[X = 1] = \frac{1}{\sqrt{2\pi}}e^{-1/2}$.
- f) T F Use the Hypergeometric Dist'n for sampling with replacement.
- g) T F If $X \sim \text{Ge}(1/10)$ then $P[X > 1] = 0.81$.
- h) T F If $X, Y \stackrel{\text{iid}}{\sim} \text{Ge}(1/10)$ then $P[X = Y] = 0.81$.
- i) T F $V[X - Y] = V[X] - V[Y]$ if X, Y are independent.
- j) T F If X is uniformly dist'd on $[0, 10]$ then $E[1/X] = 1/5$.

(Nearly) Blank Worksheet

Another Blank Worksheet

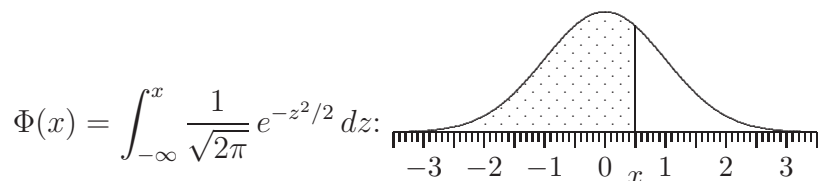


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x .

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$
 $\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2
Beta	$\text{Be}(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$x \in (0, 1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Binomial	$\text{Bi}(n, p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \dots, n$	$n p$	$n p q \quad (q = 1 - p)$
Exponential	$\text{Ex}(\lambda)$	$f(x) = \lambda e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$
Gamma	$\text{Ga}(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2
Geometric	$\text{Ge}(p)$	$f(x) = p q^x$	$x \in \mathbb{Z}_+$	q/p	$q/p^2 \quad (q = 1 - p)$
		$f(y) = p q^{y-1}$	$y \in \{1, \dots\}$	$1/p$	$q/p^2 \quad (y = x + 1)$
HyperGeo.	$\text{HG}(n, A, B)$	$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \dots, n$	$n P$	$n P (1-P) \frac{N-n}{N-1} \quad (P = \frac{A}{A+B})$
Logistic	$\text{Lo}(\mu, \beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2 \beta^2 / 3$
Log Normal	$\text{LN}(\mu, \sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x - \mu)^2 / 2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2} (e^{\sigma^2}-1)$
Neg. Binom.	$\text{NB}(\alpha, p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2 \quad (q = 1 - p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha q^{y-\alpha}$	$y \in \{\alpha, \dots\}$	α/p	$\alpha q/p^2 \quad (y = x + \alpha)$
Normal	$\text{No}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2
Pareto	$\text{Pa}(\alpha, \epsilon)$	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$
		$f(y) = \alpha \epsilon^\alpha / y^{\alpha+1}$	$y \in (\epsilon, \infty)$	$\frac{\epsilon \alpha}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2 \quad (y = x + \epsilon)$
Poisson	$\text{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ
Snedecor F	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 > 2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2 > 4$
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0 if $\nu > 1$	$\frac{\nu}{\nu-2}$ if $\nu > 2$
Uniform	$\text{Un}(a, b)$	$f(x) = \frac{1}{b-a}$	$x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Weibull	$\text{We}(\alpha, \beta)$	$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$