

## Midterm Examination # 2

STA 230 = MTH 230 = MTH 730: Probability

Wednesday, 2018 Nov 14  
10:05→11:20am

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes. **No cell phones.**
- **Show your work.** Neatness counts. Boxing answers helps.
- For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify.**
- PDF and normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: \_\_\_\_\_

Print Name Clearly: \_\_\_\_\_

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

**Problem 1:** The random variables  $X, Y$  have joint pdf  $f(x, y)$  on  $\mathbb{R}^2$ . Show what integrals (or other expressions) would be needed to find each of the following quantities, all in terms of the joint pdf  $f(x, y)$ . Be careful about the *limits* of integration.

a) (3)  $P[X > Y + 1] = \int \int$

b) (3)  $F_X(x) = \int \int$

c) (3)  $f_X(x) = \int$

d) (3)  $P[Y > X^2] = \int \int$

e) (3)  $f(x \mid y) =$

f) (3)  $P[X + Y \leq z] = \int \int$

g) (2)  $P[|X| + |Y| \leq 2] = \int \int$

**Problem 2:** The random variables  $\{U_j\} \stackrel{\text{iid}}{\sim} \text{Un}(0, 1)$  are independent with the standard uniform distribution, and  $S := \sum_{j=1}^{100} U_j$  is their sum.

a) (4) Find<sup>1</sup> the mean and variance for  $S$ :

$$E[S] = \underline{\hspace{2cm}} \qquad V[S] = \underline{\hspace{2cm}}$$

b) (4) What does the Central Limit Theorem suggest for the approximate value of:

$$P[|S - 50| > 5] \approx$$

c) (4) What bound does Markov's (or Chebychev's) Inequality give for:  
 $P[|S - 50| > 5] = P[|S - 50|^2 > 5^2] \leq$

d) (4) For constants  $c \in \mathbb{R}$ , what is:  
 $E[(S - c)^2] =$

e) (4) Find the mean  $\mu$  and variance  $\sigma^2$  for  $U^* := \max_{1 \leq j \leq 10} U_j$ .

$$\mu = \underline{\hspace{2cm}} \qquad \sigma^2 = \underline{\hspace{2cm}}$$

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<sup>1</sup>Note: See page 10 for a table of common distributions with their pdf/pmf's, CDF's, means, and variances.

**Problem 3:** The random variables  $X$  and  $Y$  have joint probability density function

$$f(x, y) = \begin{cases} cx & 0 < x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) (4) Find the normalizing constant:

$c =$  \_\_\_\_\_

b) (8) Find the marginal density functions (for *all*  $x, y$  in  $\mathbb{R}$ ):

$$f_X(x) = \left\{ \begin{array}{l} \\ \end{array} \right. \qquad f_Y(y) = \left\{ \begin{array}{l} \\ \end{array} \right.$$

c) (8) Find the pdf for  $Z := X/Y$ :

$$f_Z(z) = \left\{ \begin{array}{l} \\ \end{array} \right.$$

**Problem 4:** The random variables  $X$ ,  $Y$ , and  $Z$  are independent, with distributions and pdf/pmfs

$$\begin{aligned} X &\sim \text{Ge}(p) & Y &\sim \text{Po}(\lambda) & Z &\sim \text{Ex}(\beta) \\ f(x) &= p q^x, \ x \in \mathbb{N}_0 & g(y) &= \frac{\lambda^y}{y!} e^{-\lambda}, \ y \in \mathbb{N}_0 & h(z) &= \beta e^{-\beta z}, \ z > 0 \end{aligned}$$

where  $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ . Find (and **simplify**):

a) (4)  $\mathbf{P}[X \geq Y] =$

b) (4)  $\mathbf{P}[Z \geq X] =$

**Problem 4 (cont'd):** Again with  $X \sim \text{Ge}(p)$ ,  $Y \sim \text{Po}(\lambda)$  and  $Z \sim \text{Ex}(\beta)$  independent, find and **simplify**:

c) (4)  $P[Z \geq Y] =$

d) (4)  $P[X = Y] =$

e) (4)  $P[X = Z] =$

**Problem 5:** True or false? Circle one; each answer is worth 2 points. No explanations are needed. “iid” means Independent & Identically Distributed.

- a) T F If  $\{X_j\}$  are iid with mean  $\mu$ , then  $\frac{1}{n} \sum_{j=1}^n X_j$  converges to  $\mu$ .
- b) T F If  $X$  and  $Y$  are independent with pdfs  $f_X$  and  $f_Y$  then their product  $Z := X \cdot Y$  has pdf  $f_Z(z) = f_X(z) \cdot f_Y(z)$ .
- c) T F If  $T$  has a constant hazard function  $h(t) := f(t)/(1 - F(t))$  then  $T$  has an exponential distribution.
- d) T F If  $X \sim \text{Ex}(2)$  and  $Y := 2X$  then  $Y \sim \text{Ex}(1)$ .
- e) T F The length of time until an event has the Poisson distribution.
- f) T F  $P[X < Y] = \int_{-\infty}^y \left[ \int_x^{\infty} f(x, y) dy \right] dx$ .
- g) T F If  $X \sim \text{Ex}(\lambda)$  then  $Y := X^2$  has pdf  $f(y) = \lambda \exp(-\lambda y^2) \mathbf{1}_{\{y>0\}}$
- h) T F If  $E[XY] = E[X] E[Y]$  then  $X$  and  $Y$  are uncorrelated
- i) T F If the cond'l pdf of  $Z := X + Y$  given  $Y$  is  $f(z | y) = f_X(z - y)$  then  $X$  and  $Y$  are independent.
- j) T F The MGF for the  $\text{Un}(0, 1)$  distribution is  $M(t) \equiv 1$ .

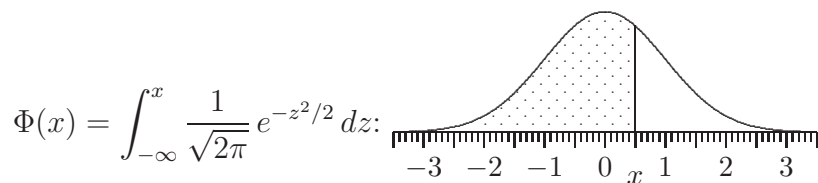


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**(Nearly) Blank Worksheet**

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## Another Blank Worksheet



**Table 5.1** Area  $\Phi(x)$  under the Standard Normal Curve to the left of  $x$ .

$x$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$     $\Phi(1.6449) = 0.95$     $\Phi(2.3263) = 0.99$     $\Phi(3.0902) = 0.999$   
 $\Phi(1.2816) = 0.90$     $\Phi(1.9600) = 0.975$     $\Phi(2.5758) = 0.995$     $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean $\mu$	Variance $\sigma^2$
Beta	$\text{Be}(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$x \in (0, 1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Binomial	$\text{Bi}(n, p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \dots, n$	$n p$	$n p q \quad (q = 1 - p)$
Exponential	$\text{Ex}(\lambda)$	$f(x) = \lambda e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$
Gamma	$\text{Ga}(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$x \in \mathbb{R}_+$	$\alpha/\lambda$	$\alpha/\lambda^2$
Geometric	$\text{Ge}(p)$	$f(x) = p q^x$	$x \in \mathbb{Z}_+$	$q/p$	$q/p^2 \quad (q = 1 - p)$
		$f(y) = p q^{y-1}$	$y \in \{1, \dots\}$	$1/p$	$q/p^2 \quad (y = x + 1)$
HyperGeo.	$\text{HG}(n, A, B)$	$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \dots, n$	$n P$	$n P (1-P) \frac{N-n}{N-1} \quad (P = \frac{A}{A+B})$
Logistic	$\text{Lo}(\mu, \beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	$\mu$	$\pi^2 \beta^2 / 3$
Log Normal	$\text{LN}(\mu, \sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x - \mu)^2 / 2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2} (e^{\sigma^2}-1)$
Neg. Binom.	$\text{NB}(\alpha, p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2 \quad (q = 1 - p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha q^{y-\alpha}$	$y \in \{\alpha, \dots\}$	$\alpha/p$	$\alpha q/p^2 \quad (y = x + \alpha)$
Normal	$\text{No}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2}$	$x \in \mathbb{R}$	$\mu$	$\sigma^2$
Pareto	$\text{Pa}(\alpha, \epsilon)$	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$
		$f(y) = \alpha \epsilon^\alpha / y^{\alpha+1}$	$y \in (\epsilon, \infty)$	$\frac{\epsilon \alpha}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2 \quad (y = x + \epsilon)$
Poisson	$\text{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	$\lambda$	$\lambda$
Snedecor $F$	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 > 2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2 > 4$
Student $t$	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0 if $\nu > 1$	$\frac{\nu}{\nu-2}$ if $\nu > 2$
Uniform	$\text{Un}(a, b)$	$f(x) = \frac{1}{b-a}$	$x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Weibull	$\text{We}(\alpha, \beta)$	$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$