Show $\bar{X}$ and $S^2$ are independent
(under the assumption the random sample is normally distributed)

A well known result in statistics is the independence of $\bar{X}$ and $S^2$ when $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$. This handout presents a proof of the result using a series of results. First, a few lemmas are presented which will allow succeeding results to follow more easily. In addition, the distribution of $\frac{(n - 1)S^2}{\sigma^2}$ is derived.

**Definition 1.** The sample variance is defined as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

**Lemma 1.** The sum of the squares of the random variables $X_1, X_2, \cdots, X_n$ is

$$\sum_{i=1}^{n} X_i^2 = (n - 1)S^2 + n\bar{X}^2$$

**Proof.** By Definition 1,

$$(n - 1)S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - 2\bar{X} \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

It follows that

$$\sum_{i=1}^{n} X_i^2 = (n - 1)S^2 + n\bar{X}^2$$

□

**Lemma 2.** The sum of squares of the random variables $X_1, X_2, \cdots, X_n$ centered about the mean, $\mu$, is

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

**Proof.** The sum of squares can be simplified as

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} X_i^2 - 2\mu \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \mu^2$$

$$= \sum_{i=1}^{n} X_i^2 - 2n\mu \bar{X} + n\mu^2$$

(1)

By Lemma 1, (1) simplifies to

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} X_i^2 - 2n\mu \bar{X} + n\mu^2 = (n - 1)S^2 + n\bar{X}^2 - 2n\mu \bar{X} + n\mu^2$$

$$= (n - 1)S^2 + n(\bar{X} - \mu)^2$$

(2)

$$= \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

□
Lemma 3. If $Z \sim N(0, 1)$, then $Z^2 \sim \chi^2(1)$.

Proof. The moment generating function for $Z^2$ is defined as

$$M_{Z^2}(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tz^2} e^{-z^2/2} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} (1 - 2t) z^2 \right] \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \frac{z^2}{1 - 2t} \right] \, dz$$

kernel of a $N(0, \frac{1}{1-2t})$

$$= \frac{1}{\sqrt{1 - 2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{1 - 2t}} \right) \exp \left[ -\frac{1}{2} \frac{z^2}{1 - 2t} \right] \, dz$$

integrates to 1

$$= \frac{1}{\sqrt{1 - 2t}}$$

$$= (1 - 2t)^{-1/2}$$

Note that this is the moment generating function for a $\chi^2$ random variable with one degree of freedom. Hence,

$$Z^2 \sim \chi^2(1)$$

Lemma 4. Suppose $X_1, X_2, \cdots, X_n$ are independent and identically distributed $\chi^2(1)$ random variables. It follows that

$$Y = \sum_{i=1}^{n} X_i \sim \chi^2(n)$$

Proof. The moment generating function of $X_i$ is

$$M_{X_i}(t) = (1 - 2t)^{-1/2}.$$

It follows that the moment generating function for $Y$ is

$$M_Y(t) = E(e^{tY}) = E[e^{tX_1 + tX_2 + \cdots + tX_n}] = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1 - 2t)^{-1/2}$$

$$= (1 - 2t)^{-\sum_{i=1}^{n} 1/2}$$

$$= (1 - 2t)^{-n/2}$$

It follows that this is the MGF for a $\chi^2$ distribution with $n$ degrees of freedom. Hence,

$$Y = \sum_{i=1}^{n} X_i \sim \chi^2(n)$$
Theorem 1. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a normal distribution with mean, $\mu$, and variance, $\sigma^2$. It follows that the sample mean, $\bar{X}$, is independent of $X_i - \bar{X}$, $i = 1, 2, \cdots, n$.

Proof. The joint distribution of $X_1, X_2, \cdots, X_n$ is

$$f_X(x_1, x_2, \cdots, x_n) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \right\}$$

Transform the random variables $X_i$, $i = 1, 2, \cdots, n$ to

$$Y_1 = \bar{X} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad X_1 = Y_1$$

$$Y_2 = X_2 - \bar{X} \quad \quad \quad \quad X_2 \quad = \quad Y_2 + Y_1$$

$$Y_3 = X_3 - \bar{X} \quad \quad \quad \quad X_3 \quad = \quad Y_3 + Y_1$$

$$\vdots \quad = \quad \vdots \quad \quad \quad \quad \quad \quad \vdots \quad = \quad \vdots$$

$$Y_n = X_n - \bar{X} \quad \quad \quad \quad X_n \quad = \quad Y_n + Y_1$$

The Jacobian of the transformation can be shown to not depend on $X_i$ or $\bar{X}$ and is equal to the constant $n$. It follows that

$$f_{Y_1,Y_2,\cdots,Y_n}(y_1, y_2, \cdots, y_n) = f_X(x_1, x_2, \cdots, x_n)|J|$$

$$= n f_X(x_1, y_1 + y_2, \cdots, y_1 + y_n)$$

$$= \text{constants} \cdot \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \right\}$$

(3)

Note that the sum in the exponent of the joint pdf can be simplified using Lemma 2. It follows that

$$\sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$= \frac{1}{\sigma^2} \left[ \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right]$$

$$= \frac{1}{\sigma^2} \left[ (x_1 - \bar{x})^2 + \sum_{i=2}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right]$$

(4)

Note that since $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$, it follows that

$$x_1 - \bar{x} = -\sum_{i=2}^{n}(x_i - \bar{x})$$
Therefore, equation (4) simplifies to
\[ \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \left[ (x_1 - \overline{x})^2 + \sum_{i=2}^{n} (x_i - \overline{x})^2 + n(\overline{x} - \mu)^2 \right] \]
\[ = \frac{1}{\sigma^2} \left[ \left( \sum_{i=2}^{n} (x_i - \overline{x}) \right)^2 + \sum_{i=2}^{n} (x_i - \overline{x})^2 + n(\overline{x} - \mu)^2 \right] \]
\[ = \frac{1}{\sigma^2} \left[ \left( \sum_{i=2}^{n} y_i \right)^2 + \sum_{i=2}^{n} y_i^2 + n(y_1 - \mu)^2 \right] \]
Therefore, the pdf of \( Y_1, Y_2, \ldots, Y_n \), equation (1), simplifies to
\[ f_{Y_1, Y_2, \ldots, Y_n}(y_1, y_2, \ldots, y_n) = \text{constants} \cdot \exp \left\{ \frac{-1}{2} \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \right\} \]
\[ = \text{constants} \cdot \exp \left\{ \frac{-1}{2\sigma^2} \left[ \left( \sum_{i=2}^{n} y_i \right)^2 + \sum_{i=2}^{n} y_i^2 + n(y_1 - \mu)^2 \right] \right\} \]
\[ = \left[ \exp \left\{ \frac{-n}{2\sigma^2} (y_1 - \mu)^2 \right\} \right] \left. \frac{h(y_2, y_3, \ldots, y_n)}{g(y_1)} \right| \]

Because \( f_{Y_1, Y_2, \ldots, Y_n}(y_1, y_2, \ldots, y_n) \) can be factored into a product of functions that depend only their respective set of statistics, it follows that \( Y_1 = \overline{X} \) is independent of \( Y_i = X_i - \overline{X}, \ i = 2, 3, \ldots, n \).

Finally, since \( X_1 - \overline{X} = -\sum_{i=2}^{n} (X_i - \overline{X}) \), it follows that \( X_1 - \overline{X} \) is a function of \( X_i - \overline{X}, \ i = 2, 3, \ldots, n \). Therefore, \( X_1 - \overline{X} \) is independent of \( Y_1 = \overline{X} \).

**Theorem 2.** Suppose \( X_1, X_2, \ldots, X_n \) is a random sample from a normal distribution with mean, \( \mu \), and variance, \( \sigma^2 \). It follows that the sample mean, \( \overline{X} \), is independent of the sample variance, \( S^2 \).

**Proof.** The definition of \( S^2 \) is given in Definition 1. Because \( S^2 \) is a function of \( X_i - \overline{X}, \ i = 1, 2, \ldots, n \), it follows that \( S^2 \) is independent of \( \overline{X} \).

**Theorem 3.** Suppose \( X_1, X_2, \ldots, X_n \) is a random sample from a normal distribution with mean, \( \mu \), and variance, \( \sigma^2 \). It follows that the distribution of a multiple of the sample variance follows a \( \chi^2 \) distribution with \( n - 1 \) degrees of freedom. In particular,
\[ \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \]

**Proof.** Equation (2) states
\[ \sum_{i=1}^{n} (X_i - \mu)^2 = (n-1)S^2 + n(\overline{X} - \mu)^2. \]
It follows that
\[
\sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 = \frac{(n-1)S^2}{\sigma^2} + n \left( \frac{\bar{X} - \mu}{\sigma} \right)^2
\]
\[
\sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2
\]

\[U = W + V\]

Note that since \(X_i \sim \mathcal{N}(\mu, \sigma^2)\), it follows that \(Z_i = \frac{X_i - \mu}{\sigma} \sim \mathcal{N}(0,1)\). Similarly, since \(\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})\), then \(\bar{X} - \mu \sim \mathcal{N}(0,1)\). By Lemma 3, it follows that \(\left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(1)\) and \(V = \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \sim \chi^2(1)\).

By Lemma 4, it follows that \(U = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)\). Therefore, since \(W\) and \(V\) are independent, then the moment generating function of \(U\) is

\[M_U(t) = M_W(t)M_V(t)\]

\[(1 - 2t)^{-n/2} = M_W(t)(1 - 2t)^{-1/2}\]

\[\Rightarrow M_W(t) = \frac{(1 - 2t)^{-n/2}}{(1 - 2t)^{-1/2}} = (1 - 2t)^{-(n-1)/2}\]

The moment generating function for \(W\) is recognized as coming from a \(\chi^2\) distribution with \(n-1\) degrees of freedom. Hence,

\[W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)\]

\(\square\)