Problem 1: Recall that a sequence \( \{x_n\} \) in a metric space \((X, d)\) converges to a limit \( x^* \in X \) if for each \( \epsilon > 0 \) there exists a number \( N_\epsilon < \infty \) such that
\[
(\forall n \geq N_\epsilon) \ d(x_n, x^*) < \epsilon.
\]

a. Prove\(^1\) that \( x_n := 1/\sqrt{n} \) converges to \( x^* = 0 \) in the metric space \( X = \mathbb{R} \) with the usual (Euclidean) distance metric \( d(x, y) := |x - y| = \sqrt{(x - y)^2} \).

b. Find an explicit sequence \( x_n \) of rational numbers that converges to \( x^* = \pi \) in the metric space \( X = \mathbb{R} \). Prove that it converges, by finding \( N_\epsilon \) (Hint: you might want to start by choosing \( N_\epsilon \)— say, \( [1/\epsilon] \) or \( \lceil-\log_2 \epsilon\rceil \) or \( \lceil-\log_{10} \epsilon\rceil \) — and then find \( x_n \)).

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\(^1\)Find \( N_\epsilon \) explicitly. You may find the function \( \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\} \) (the greatest integer less than or equal to \( x \)) to be useful, or perhaps \( \lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\} \).
Problem 2: Recall that a subset $E$ of a metric space $(X, d)$ is open if for each $x \in E$ there exists some $\epsilon_x > 0$ such that the entire ball

$$B_\epsilon(x) = \{ \xi \in X : d(x, \xi) < \epsilon_x \} \subset E$$

and that a set $F \subset X$ is closed if its complement $F^c = \{ x \in X : x \notin F \}$ is open.

a. Prove that $(0, 1)$ is open in $X = \mathbb{R}$.

b. Prove that any union $U = \bigcup E_\alpha$ of open sets is also open.

c. Show by example that the union $U = \bigcup F_\alpha$ of closed sets may not be closed.
Problem 3: Recall that a set $K$ in a metric space $(\mathcal{X}, d)$ is compact\(^2\) if every open cover $K \subseteq \bigcup_{\alpha} U_{\alpha}$ admits a finite sub-cover $K \subseteq \bigcup_{i=1}^{n} U_{a_{i}}$, and that a function $f(\cdot) : \mathcal{X} \to \mathcal{Y}$ from one metric space to another is continuous if for every open set $U \subset \mathcal{Y}$, $f^{-1}(U) := \{x : f(x) \in U\}$ is an open set in $\mathcal{X}$.

a. Prove that every compact set $K$ is also closed.

b. If $K$ is a compact set and $A \subset K$ is a closed subset, prove that $A$ is also compact.

c. If $f : \mathcal{X} \to \mathbb{R}$ is a continuous real-valued function and $K \subset \mathcal{X}$ is compact, prove that the supremum

$$M := \sup_{x \in K} f(x)$$

is finite.

d. Show\(^3\) this can fail if $f$ is not continuous—i.e., give an example of an unbounded (but finite) function $f$ on a compact set $K$.

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\(^2\)The Heine-Borel theorem says in Euclidean space any closed & bounded set is compact, but that doesn’t hold in general. For example, the unit ball $B := \{f : \int_{0}^{1} |f(x)|^2 \, dx \leq 1\}$ is closed and bounded in $L_2([0,1])$ but is not compact, since the sequence of functions $\{f_n(x) := \sqrt{2}\sin(n\pi x)\} \subset B$ has no limit point in $B$.

\(^3\)Suggestion: take $K = [0,1]$ on $\mathcal{X} = \mathbb{R}$, and define $f(x)$ by cases. What cases?
Problem 4:

a. Let \( K_\alpha \) be compact for each index \( \alpha \) and suppose that each finite intersection \( \bigcap_{j=1}^{n} K_{\alpha_j} \neq \emptyset \) is non-empty. Prove that \( \bigcap_{\alpha} K_\alpha \neq \emptyset \).

b. If \( f : \mathcal{X} \to \mathbb{R} \) is real-valued and continuous with supremum \( M := \sup_{x \in K} f(x) \) on a compact set \( K \subset \mathcal{X} \), prove that there exists some \( x^* \in K \) for which \( f(x^*) = M \).
Problem 5:

a. Give an example of a closed set $C \subset \mathbb{R}$ that is *not* compact.

b. Give an example of a set $A \subset \mathbb{R}$ that is neither closed nor open.

c. Give an example of a set $B \subset \mathbb{R}$ that is both closed and open.
Problem 6: Evaluate the sums and integrals below for every value of $p \in \mathbb{R}$ (some expressions might be infinite or undefined for some values of $p$):

a. $\int_{0}^{1} x^p \, dx =$

b. $\int_{0}^{\infty} e^{-px} \, dx =$

c. $\sum_{n=2}^{9} p^n =$

d. $\sum_{n=1}^{\infty} p^n =$

e. $\sum_{n=7}^{\infty} n p^n =$

f. $\int_{0}^{\infty} x e^{-px^2} \, dx =$

g. $\int_{0}^{e} \sin(\ln u) \, du =$

h. $\int_{0}^{\pi} e^{-p \cos(x)} \sin(x) \, dx =$
Problem 7: Which of the following sums and integrals converges (to a finite limit)? Why? You need not evaluate the limit.

a. T F \( \int_2^\infty \frac{\ln(e^x - 2)}{x^3 + 1} \, dx \) converges:

b. T F \( \sum_{n=0}^{\infty} \frac{3^n(n!)^2}{(2n)!} \) converges:

c. T F \( \sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^{3/2}} \) converges:

d. T F \( \int_0^\infty \frac{\sin x}{x^{3/2}} \, dx \) converges:

e. T F \( \int_0^\infty \frac{dx}{\sqrt{x} + x^2} \) converges:

f. T F \( \int_0^4 \frac{\tan x}{x^2} \, dx \) converges: