Midterm Examination I

STA 711: Probability & Measure Theory

Wednesday, 2015 Sep 30, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, give answers in **closed form** (without any unevaluated sums, integrals, maxima, unreduced fractions, *etc.*) where possible and **simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\{A_i\}$ be a countable collection of events on some probablity space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) (10) Set $B_n := \cup_{i=1}^n A_i$ and $B := \cup_{i=1}^\infty A_i$. Does $\mathsf{P}[B_n] \to \mathsf{P}[B]$? Give a proof or a counter-example.

Problem 1 (cont'd): Still $\{A_i\} \subset \mathcal{F}$ on $(\Omega, \mathcal{F}, \mathsf{P})$.

b) (10) Let ω and ω' be two outcomes in Ω that are not "separated" by $\{A_i\}$, *i.e.*, such that for each i either $\{\omega, \omega'\} \subset A_i$ or $\{\omega, \omega'\} \subset A_i^c$. Prove that the σ-algebra $\mathcal{A} := \sigma(\{A_i\})$ doesn't separate ω and ω' either.

Problem 2: Let $\{X_k\} \sim \mathsf{Ex}(1)$ be random variables, each with the unit exponential distribution— so $\mathsf{P}[X_k \in B] = \int_B e^{-x}\mathbf{1}_{\{x>0\}}\,dx$ for $B \in \mathcal{B}(\mathbb{R})$. Set $S_n := \sum_{k=1}^n \mathbf{1}_{\{X_k>k\}}$ and $T_n := \sum_{k=1}^n \mathbf{1}_{\{X_k>1\}}$.

a) (4) Evaluate these expectations. Simplify!

$\mathsf{E}[S_n] =$ $\mathsf{E}[T_n] =$

b) (4) Does $\{S_n\}$ converge to a finite random variable as $n \to \infty$? ◯ Yes ◯ No Why? If this can't be determined from the information given in the problem (read it carefully), explain.

c) (4) Is $\{S_n\}$ dominated by an L_1 random variable Y? ◯ Yes ◯ No If so, give Y explicitly and evaluate $\mathsf{E}[Y]$ (Simplify!); if not, say why.

Problem 2 (cont'd): Still $\{X_k\} \sim \mathsf{Ex}(1)$ for each $k \in \mathbb{N}$, $S_n := \sum_{k=1}^n \mathbf{1}_{\{X_k > k\}}$, and $T_n := \sum_{k=1}^n \mathbf{1}_{\{X_k > 1\}}$.

d) (4) Does $\{\mathsf{P}[T_n = 0]\}$ converge to zero as $n \to \infty$? ◯ Yes ◯ No
Why? If this can't be determined from the information given in the problem (read it carefully), explain.

e) (4) Is $\{T_n\}$ dominated by an L_1 random variable Z? ◯ Yes ◯ No
If so, give Z explicitly and evaluate $\mathsf{E}[Z]$ (Simplify!); if not, say why. If this can't be determined from the information given in the problem, explain.

Problem 3: A space $\mathcal{X}$ is called "sigma-compact" if it can be written as the countable union $\mathcal{X} = \cup K_j$ of (not necessarily disjoint) compact sets. Recall that in $\mathbb{R}^n$ a set K is compact if and only if it is closed and bounded.

a) (5) Verify that $\mathbb{R}$ is sigma-compact (+1XC to show $\mathbb{R}^n$ σ-cpt too).

b) (10) Let P be any probability measure on the Borel sets $\mathcal{B}$ of any sigma-compact space $\mathcal{X}$. Show that for any $\epsilon > 0$ there is some compact set $K \subset \mathcal{X}$ with $\mathsf{P}(K) > (1 - \epsilon)$.

c) (5) A collection of probability measures $\{\mathsf{P}_j\}$ on the Borel sets of some space $\mathcal{X}$ is called "tight" if for any $\epsilon > 0$ there exists a single compact set $K \subset \mathcal{X}$ such that $\mathsf{P}_j(K) > (1 - \epsilon)$ for every P_j. Prove that any **finite** collection of $n < \infty$ probability measures on any σ-compact space $\mathcal{X}$ is tight.

Problem 4: Let $\Omega = (0, 1]$ with Borel sets $\mathcal{F}$ and Lebesgue measure P.

a) (10) Find a sequence of L_∞ random variables X_n with $\|X_n\|_\infty < \infty$ for each $n \in \mathbb{N}$ and $X_n(\omega) \to 0$ for each $\omega \in \Omega$ but $\mathsf{E}[X_n] \not\to 0$, if possible; if this is not possible, explain why.

b) (10) Find a sequence of L_∞ random variables X_n with $\|X_n\|_\infty < 2$ for each $n \in \mathbb{N}$ and $X_n(\omega) \to 0$ for each $\omega \in \Omega$ but $\mathsf{E}[X_n] \not\to 0$, if possible; if this is not possible, explain why.

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F Lebesgue's DCT implies that $\mathsf{E}\big[\sin(n\pi U)\big] \to 0$ as $n \to \infty$ for $U \sim \mathsf{Un}(0,1)$.

b) T F The subsets of $\Omega = \{1, 2, \cdots, 10\}$ with an even number of elements form a λ-system.

c) T F The subsets of $\Omega = \{1, 2, \cdots, 10\}$ with an even number of elements form a π-system.

d) T F Each π-system on any Ω includes $\emptyset$.

e) T F $(\forall t \in \mathbb{R})$, the MGF $M(t) := \mathsf{E}\big[\exp(tX)\big] \geq 1 + t\mathsf{E}[X]$.

f) T F If $\{A_n\}$ are independent, then $\{A_n^c\}$ are independent too.

g) T F If $\{A_n\}$ are independent and each is independent of B, then $\{A_n \cap B\}$ are independent too.

h) T F Every convex function ϕ on $\mathbb{R}$ is continuous.

i) T F Let $X \in L_1$ and $Y := \exp(X)$. Then $\log(\mathsf{E}[Y]) \leq \mathsf{E}[X]$.

j) T F $\{X_n := n\mathbf{1}_{\{(0,1/n^2]\}}\}$ on $(0, 1]$ (w/Borel sets & Lebesgue measure) are dominated by some $Y \in L_1$.

Blank Worksheet

Another Blank Worksheet

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	