Midterm Examination I

STA 711: Probability & Measure Theory

Wednesday, 2016 Oct 05, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, give answers in **closed form** (without any unevaluated sums, integrals, maxima, unreduced fractions, *etc.*) where possible and **simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: The "moment generating function" (MGF) for a real-valued random variable X is the positive real-valued function

$$M_X(t) := \mathsf{E}\big[\exp(tX)\big],$$

defined for all $t \in \mathbb{R}$ (but possibly equal to $+\infty$ for some t).

a) (10) Show that, for every $x \in \mathbb{R}$,

$$\mathsf{P}[X \geq x] \leq M_X(2)e^{-2x}$$

b) (5) If $M_X(1) + M_X(-1) < \infty$, show that $\mathsf{E}|X| < \infty$.

c) (5) If $\mathsf{E}|X| < \infty$, show that

$$\mathsf{E}X \leq \log M_X(1)$$

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be $\Omega = (0, 1]$ with the Borel sets $\mathcal{F}$ and Lebesgue measure P. For each $n \in \mathbb{N}$ let

$$\mathcal{F}_n := \left\{ \cup_j \left(a_j 2^{-n}, b_j 2^{-n}\right] \right\}$$

be the σ-field generated by left-open sets of the form $(0, b/2^n]$ for non-negative integers $b \le 2^n$. Set $Y(\omega) := 1/\omega$, and let X_n be the largest $\mathcal{F}_n$-measurable random variable with $X_n \le Y$.

a) (5) Find $X_1(\omega)$ for each $\omega \in \Omega$.
$X_1(\omega) =$

b) (5) Find $\mathsf{E}X_1$ for your X_1 from part a) above.
$\mathsf{E}X_1 =$

c) (5) Find the largest $\mathcal{F}_2$ random variable $X_2 \le Y$.
$X_2(\omega) =$

d) (5) What is the limit $\lim_{n\to\infty} \mathsf{E}X_n$? Why?

Problem 3: Let $\Omega = (0, 1]$ with the Borel sets $\mathcal{F}$, and let P be a probability measure on $(\Omega, \mathcal{F})$ with the property that $\mathsf{P}[\{\omega\}] = 0$ for each $\omega \in \Omega$. Fix $\epsilon > 0$.

a) (10) For any point $x \in (0, 1)$, show there exists an open interval $V \subset \Omega$ with $x \in V$ and $\mathsf{P}[V] \leq \epsilon$.

b) (10) Show that for any countable dense set $\{x_i\} \subset (0, 1)$ there exists an open set $V \subset \Omega$ such that $\{x_i\} \subset V$ and $\mathsf{P}[V] \leq \epsilon$.

Problem 4: Let $\Omega = \{1, 2, 3\}$ with $\mathcal{F} = 2^\Omega$ and P determined by $\mathsf{P}[\{\omega\}] = \omega/6$. Define random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ by

$$X(\omega) := \omega \qquad Y(\omega); = 1/\omega$$

a) (5) Find $\|X\|_p$ for $p = 1$, $p = 2$, and $p = \infty$. Simplify!

b) (5) Find $\|Y\|_p$ for $p = 1$, $p = 2$, and $p = \infty$. Simplify!

c) (5) Verify that $\mathsf{E}[XY] < \mathsf{E}[X]\,\mathsf{E}[Y]$.

Problem 4 (cont'd): Still $X(\omega) > 0$ and $Y = 1/X$, but now $(\Omega, \mathcal{F}, \mathsf{P})$ is arbitrary.

d) (5) For any positive non-constant random variable $X > 0$ on any probability space $(\Omega, \mathcal{F}, \mathsf{P})$, set $Y := 1/X$ and prove that $\mathsf{E}[X]\,\mathsf{E}[Y] > \mathsf{E}[XY] = 1$.

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$. The notation "$A \perp\!\!\!\perp B$" means that A and B are independent, for events or RVs.

a) T F If $\mathsf{P}[N] = 0$ then $N \perp\!\!\!\perp A$ for every $A \in \mathcal{F}$.

b) T F The subsets of $\Omega = \{1, 2, \cdots, 9\}$ with an even number of elements form a λ-system.

c) T F The subsets of $\Omega = \{1, 2, \cdots, 9\}$ with an even number of elements form a π-system.

d) T F If events $A \perp\!\!\!\perp B$ then RVs $\mathbf{1}_A \perp\!\!\!\perp \mathbf{1}_B$.

e) T F If $A \cap B = \emptyset$ then A and B are independent.

f) T F If $B_n \supset B_{n+1}$ for every n, then $\mathsf{P}[B_n] \to 0$.

g) T F If $\mathsf{P}[A] + \mathsf{P}[B] > 1$ then $A \cap B \neq \emptyset$.

h) T F If $f : \mathbb{R} \to \mathbb{R}_+$ is continuous and monotone increasing, then $g(x) := \frac{f(x)}{1+f(x)}$ is *uniformly* continuous.

i) T F If X and $1/X$ are independent, then X is constant *a.s.*

j) T F For any $B \in \mathcal{F}$, $\{B\}$ is a π-system on $(\Omega, \mathcal{F}, \mathsf{P})$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}{}^*$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}{}^*$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0^*	$\nu/(\nu-2)^*$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	