Midterm Examination I

STA 711: Probability & Measure Theory

Wednesday, 2017 Oct 04, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\Omega := \{1, 2, \ldots, 100\}$ be the integers from one to 100, and let $\mathcal{C} := \big\{\{i,j\} : \ i,j \in \Omega, i < j\big\}$ be all subsets of two distinct elements. Set $A := \{1, \ldots, 50\}$ and $B := \{42\}$.

a) (5) Describe[1] the π-system $\pi(\mathcal{C})$ determined by $\mathcal{C}$, and answer (by checking): Is $A \in \pi(\mathcal{C})$? ◯ Yes ◯ No Is $B \in \pi(\mathcal{C})$? ◯ Yes ◯ No .

b) (5) Describe the λ-system $\lambda(\mathcal{C})$. Is $A \in \lambda(\mathcal{C})$? ◯ Yes ◯ No
Is $B \in \lambda(\mathcal{C})$? ◯ Yes ◯ No .

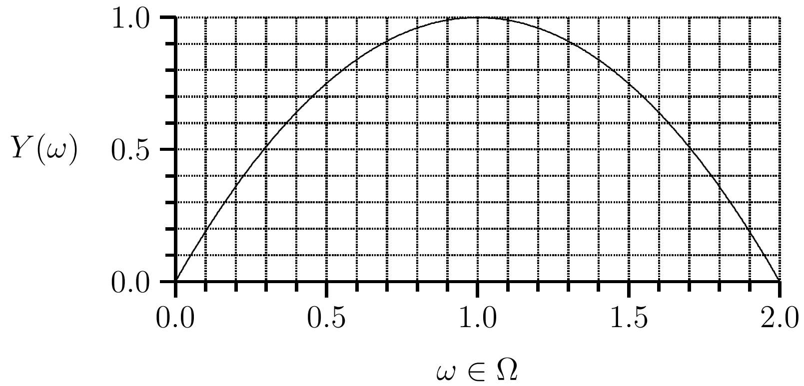
c) (5) Describe the field $\mathcal{F}(\mathcal{C})$. Is $A \in \mathcal{F}(\mathcal{C})$? ◯ Yes ◯ No
Is $B \in \mathcal{F}(\mathcal{C})$? ◯ Yes ◯ No .

d) (5) Describe the σ-field $\sigma(\mathcal{C})$. How many elements does it have?

[1]Show understanding, don't just give the definition. How can you tell if an event is in $\pi(\mathcal{C})$ or not?

Problem 2: Let $Y := \omega(2-\omega)$ be a random variable on the space $\Omega := (0,2]$ with $\mathcal{F} := \mathcal{B}(\Omega)$, the Borel sets ($Y$ is plotted below).

a) (8) Find (2pts) *and plot* (6 pts) a non-negative *simple* random variable $X \in \mathcal{E}_+$ satisfying $0 \le X(\omega) \le Y(\omega)$ and $|Y(\omega) - X(\omega)| \le 0.4$ for all $\omega \in \Omega$.



$X(\omega) =$

b) (6) Find $\mathsf{E}X$ and $\mathsf{E}Y$ for the probability measure $\mathsf{P}(d\omega) := d\omega/2$ (*i.e.*, $\mathsf{P}\{(a,b]\} = (b-a)/2$ for all $0 \le a \le b \le 2$):

$\mathsf{E}X =$ ____________ $\mathsf{E}Y =$ ____________

c) (6) Let $Z := 1_{(0,1]}(\omega)$. Are Y and Z independent on $(\Omega, \mathcal{F}, \mathsf{P})$?
◯ Yes ◯ No Why?

Problem 3: Let $\Omega := \{a, b, c, d\}$ with $\mathcal{F} := 2^\Omega$ and P that assigns probabilities 0.20, 0.60, and 0.05 respectively to the singleton sets $\{a\}$, $\{b\}$ and $\{c\}$. Consider the two fields

$$\mathcal{C}_1 := \big\{\emptyset,\ \{a, b\},\ \{c, d\},\ \Omega\big\}$$
$$\mathcal{C}_2 := \big\{\emptyset,\ \{a, c\},\ \{b, d\},\ \Omega\big\}$$

a) (8) Are $\mathcal{C}_1$ and $\mathcal{C}_2$ independent? Give a proof or counterexample.
Y N Why?

b) (6) Find a real random variable X that is $\mathcal{C}_1\backslash\mathcal{B}$-measurable but *not* $\mathcal{C}_2\backslash\mathcal{B}$-measurable (be careful not to mix up 1 and 2).

$X(a) =$ ____ $X(b) =$ ____ $X(c) =$ ____ $X(d) =$ ____

c) (6) Find all random variables that are *both* $\mathcal{C}_1\backslash\mathcal{B}$ *and* $\mathcal{C}_2\backslash\mathcal{B}$-measurable. Justify your answer.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the nonnegative integers $\Omega = \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ with $\mathcal{F} = 2^\Omega$ and $\mathsf{P}[A] := e^{-2} \sum_{\omega \in A} (2^\omega / \omega\,!)$ for $A \in \mathcal{F}$.

a) (7) Fix $p > 0$. Is the random variable $X(\omega) := 2^\omega$ in $L_p(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\|X\|_p$ in closed form. If not, tell why. If this depends on p, explain.
◯ Yes ◯ No ◯ It Depends Reasoning?

$\|X\|_p =$ ____________________

b) (6) Is $Z(\omega) := \omega$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\mathsf{E}Z$ (a numerical answer). If not, explain. ◯ Yes ◯ No Reasoning:

$\mathsf{E}Z =$ ____________________

c) (7) For $n \in \mathbb{N}$ define a random variable Y_n by $Y_n(\omega) = n$ if $\omega \geq n$, $Y_n(\omega) = 0$ if $\omega < n$. Does the Dominated Convergence Theorem apply to $\{Y_n\}$? If so, tell what DCT says and show why it applies; if not, explain why.
◯ Yes ◯ No Reasoning:

d) (XC) Find $\mathsf{E}[Z^2]$ (see b)).

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$. The notation "$A \perp\!\!\!\perp B$" means that A and B are independent, for events or RVs.

a) T F There does not exist a field $\mathcal{F}$ with exactly 42 elements.

b) T F If $A_n \subset A_{n+1}$ for every n, then $\mathsf{P}[A_n] \to 1$.

c) T F If $\mathsf{P}[A] < \mathsf{P}[B]$ and $X \geq 0$ then $\mathsf{E}[X\mathbf{1}_A] \leq \mathsf{E}[X\mathbf{1}_B]$.

d) T F If $|X| \leq Y^2$ and $Y \in L_4$ then $X \in L_8$.

e) T F If $\mathsf{P}[A] + \mathsf{P}[B] + P[C] > 1$ then $A \cap B \cap C \neq \emptyset$.

f) T F If $X \perp\!\!\!\perp Y$ then $X^{-1}(A) \perp\!\!\!\perp Y^{-1}(B)$ for all $A, B \in \mathcal{B}(\mathbb{R})$.

g) T F The finite subsets of $\Omega = \mathbb{N}$ form a π-system.

h) T F If $A \in \mathcal{F}$ then $\mathcal{G} := \{B \in \mathcal{F} : A \perp\!\!\!\perp B\}$ is a λ-system.

i) T F If X and X^2 are independent, then X is constant *a.s.*

j) T F If $A \notin \mathcal{F}$ and $B \in \mathcal{F}$ then $(A \cup B) \notin \mathcal{F}$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}{}^*$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}{}^*$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0^*	$\nu/(\nu-2)^*$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	