Final Examination

STA 711: Probability & Measure Theory

Monday, 2018 Dec 17, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible **simplify**.

Good luck.

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Print Name: ______________________________

Problem 1: Let $\{A_n\} \subset \mathcal{F}$ be independent events with probabilities $\mathsf{P}[A_n] = 1/n$, and let $X_n := \mathbf{1}_{A_n}$ be their indicator RVs.

a) (5) Does $\sum_n X_n$ converge *a.s.* to an $\mathbb{R}$-valued limit X? ◯ Yes ◯ No
Why?

b) (5) Does $\sum_n X_{n^2}$ converge *a.s.* to an $\mathbb{R}$-valued limit X? ◯ Yes ◯ No
Why?

c) (5) Does $\sum_n X_{n^2}$ converge in L_1 to an $\mathbb{R}$-valued limit X? ◯ Yes ◯ No
Why?

d) (5) Does $\sum_n n\, X_{2^n}$ converge in L_p to an $\mathbb{R}$-valued limit X for each $0 < p < \infty$? ◯ Yes ◯ No Why?

Problem 2: Let $\{X_n\}$ and Y be real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $X_n \to Y$ *a.s.* For each $n \in \mathbb{N}$, $\mathsf{E}[X_n^2] \le 100$.

a) (5) Does it follow that $Y \in L_2$? ◯ Yes ◯ No Why?

b) (5) Does it follow that $X_n \to Y$ in L_2? ◯ Yes ◯ No
Proof or counter-example:

c) (5) Is $\mathsf{P}\big[|X_n - Y| > \epsilon\big]$ summable for each $\epsilon > 0$? ◯ Yes ◯ No
Proof or counter-example:

d) (5) Is $\mathsf{P}\big[|X_1 - Y|^2 > n\epsilon\big]$ summable for each $\epsilon > 0$? ◯ Yes ◯ No
Proof or counter-example:

Problem 3: Let $X \sim \mathsf{Ex}(\lambda)$ and $Y \sim \mathsf{Ge}(p)$ be independent, with pdf $f(x) = \lambda e^{-\lambda x}\mathbf{1}_{\{x>0\}}$ and pmf $p(y) = p\,q^y, \quad y \in \mathbb{N}_0$, respectively, where $q := 1-p$.

a) (5) Find $\mathsf{P}[Y > X] =$

b) (5) Is the distribution $\mu(dz)$ of $Z := X+Y$ ◯ Absolutely Continuous, ◯ Discrete, or ◯ Neither? Give its survival function at *all* $z \in \mathbb{R}$.
$\bar{F}(z) := \mathsf{P}[Z > z] =$

Problem 3 (cont'd): Still $X \sim \mathsf{Ex}(\lambda) \perp\!\!\!\perp Y \sim \mathsf{Ge}(p)$ and $Z := X + Y$.

c) (6) Find the characteristic functions of all three RVs:

$\chi_X(\omega) =$

$\chi_Y(\omega) =$

$\chi_Z(\omega) =$

d) (4) Find the indicated conditional expectation:
$\mathsf{E}[Z \mid X] =$

Problem 4: Let $Z \sim \mathsf{No}(0,1)$ and set $X := (Z \vee 0)$, the maximum of Z and zero.

a) (5) Is the distribution $\mu(dx)$ of $X := (Z \vee 0)$ ◯ Absolutely Continuous, ◯ Discrete, or ◯ Neither? Give its survival function at *all* $x \in \mathbb{R}$., or some other representation of its distribution.
$\bar{F}(x) := \mathsf{P}[X > x] =$

b) (5) Find the moment generating function (MGF) of X. Your expression may include the normal CDF $\Phi(\cdot)$.
$M(t) := \mathsf{E}[e^{tX}] =$

c) (5) Find the mean of X (use any method you like).
$\mathsf{E}[X] =$

d) (5) Every MGF satisfies $M(0) = 1$. Is there any other $t^* \neq 0$ for which *this* $M(t^*) = 1$? Why, or why not?

Problem 5: Let $\{\xi_n\} \sim \mathsf{Po}(n^2)$.

a) (5) Find the log ch.f.[1] for $X_n := \xi_n/n^2$:
$\phi_n(\omega) = \log \mathsf{E}\big[e^{i\omega X_n}\big] =$

b) (5) Show that $\phi_n(\omega)$ converges as $n \to \infty$, and find the limit $\phi(\omega)$. What distribution has ch.f. $\exp\big(\phi(\omega)\big)$?

[1]Suggestion: First compute the ch.f. $\phi(\theta) := \mathsf{E}[e^{i\theta X}]$ for $X \sim \mathsf{Po}(\lambda)$.

Problem 5 (cont'd): Still $\{\xi_n\} \sim \mathsf{Po}(n^2)$.

c) (5) Find the log ch.f. for $Y_n := (\xi_n/n) - n$:
$\psi_n(\omega) =$

d) (5) Show that $\psi_n(\omega)$ converges as $n \to \infty$, and find the limit $\psi(\omega)$. Identify the limiting distribution of $\{Y_n\}$, which has ch.f. $\exp\big(\psi(\omega)\big)$.

Problem 6: Let $X_0{:=}1$ and, for $n \in \mathbb{N}$, let $X_n{=}2X_{n-1}$ or $X_n{=}0$ with probability 1/2 each. Set $\tau{:=}\inf\{n : X_n = 0\}$ and $\mathcal{F}_n := \sigma\{X_j : 1{\le}j{\le}n\}$.

a) (6) Prove that $(X_n, \mathcal{F}_n)$ is a martingale (reminder: there are *two* conditions to verify).

b) (4) For each $p > 0$: is $\{X_n\}$ uniformly bounded in L_p? If so, by what?

c) (4) Does $\{X_n\}$ converge to some limit X_∞ as $n \to \infty$? If so, to what limit, and in what sense(s)? If not, why not?

d) (4) Is τ in L_1? Prove it (and find $\mathsf{E}[\tau]$) or disprove it.

e) (2) Find:

$\mathsf{E}[X_\tau] =$ $\mathsf{E}[X_{\tau\wedge 10}] =$

Problem 7: Let A, B, C be independent with probabilities a, b, c, respectively on $(\Omega, \mathcal{F}, \mathsf{P})$. Find:

a) (5) $\mathsf{P}[A \cup B] =$

b) (5) $\mathsf{P}[A \cup B \mid B \cup C] =$

c) (5) $\mathsf{P}[A \cup B \cup C] =$

d) (5) $\mathsf{P}[A \mid A \cup B \cup C] =$

Problem 8: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, \mathsf{P})$; ϕ, ψ are arbitrary Borel functions on $\mathbb{R}$.

a) T F If $X_n \to X$ *a.s.* then $\liminf_{n\to\infty} X_n = X$.

b) T F If $X = \phi(Z)$ and $Y = \psi(Z)$ then X, Y can't be independent.

c) T F If $g(\cdot)$ is continuous and $X_n \to X$ $(pr.)$ then $g(X_n) \to g(X)$ $(pr.)$.

d) T F If $X \perp\!\!\!\perp Y$ and ϕ, ψ are bounded functions $\mathbb{R} \to \mathbb{R}$ then $\mathsf{E}\big[\exp\big(\phi(X) + \psi(Y)\big)\big] = \mathsf{E}\big[\exp\big(\phi(X)\big)\big] \cdot \mathsf{E}\big[\exp\big(\psi(Y)\big)\big]$.

e) T F If $A, B \in \mathcal{F}$ then $\sigma\{A, B\} = \sigma\{\mathbf{1}_A + 2\mathbf{1}_B\}$.

f) T F If $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ then

$$\mathsf{E}\big[\mathsf{E}[X \mid \mathcal{H}] \bigm| \mathcal{G}\big] = \mathsf{E}[X \mid \mathcal{G}]$$

g) T F If $\emptyset \neq \Lambda_1 \subsetneq \Lambda_2 \subsetneq \cdots \subsetneq \Lambda_n = \Omega$, then $\sigma\{\Lambda_j : 1 \le j \le n\}$ has 2^n elements.

h) T F If probability measures P, Q agree on a field $\mathcal{G}_0$ then they agree on the σ-field $\mathcal{G} = \sigma(\mathcal{G}_0) \subset \mathcal{F}$ it generates.

i) T F If $0 \le X \in L_1$ then $Y := \log(1 + X)$ satisfies $Y \in L_1$.

j) T F If each $X_j \in L_{p_j}$ for some $\{p_j\} \subset \mathbb{R}_+$ and if $\sum p_j < \infty$ then $X_+ := \sum X_j$ converges in L_1.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
		$x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$				
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	