STA 711: Probability and Measure Theory
Analysis & Calculus Quiz

Students in STA 711: Probability & Measure Theory are expected to be familiar with real analysis at an advanced undergraduate level— the level of W. Rudin’s *Principles of Mathematical Analysis* or M. Reed’s *Fundamental Ideas of Analysis*. They should be able to answer the questions in this quiz without consulting reference materials.

**Problem 1:** Recall that a sequence \( \{x_n\} \) in a metric space \((X, d)\) converges to a limit \(x^* \in X\) if for each \(\epsilon > 0\) there exists a number \(N_\epsilon < \infty\) such that

\[
(\forall n \geq N_\epsilon) \quad d(x_n, x^*) < \epsilon.
\]

a. Prove\(^1\) that \(x_n := \frac{1}{\sqrt{n}}\) converges to \(x^* = 0\) in the metric space \(X = \mathbb{R}\) with the usual (Euclidean) distance metric \(d(x, y) := |x - y| = \sqrt{(x - y)^2}\).

b. Find an explicit sequence \(x_n\) of rational numbers that converges to \(x^* = \pi\) in the metric space \(X = \mathbb{R}\). Prove that it converges, by finding \(N_\epsilon\) (Hint: you might want to start by choosing \(N_\epsilon\) — say, \(\lceil 1/\epsilon \rceil\) or \(\lceil -\log_2 \epsilon \rceil\) or \(\lceil -\log_{10} \epsilon \rceil\) — and then find \(x_n\)).

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\(^1\)Find \(N_\epsilon\) explicitly. You may find the function \(\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}\) (the greatest integer less than or equal to \(x\)) to be useful, or perhaps \(\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}\).
Problem 2: Recall that a subset $E$ of a metric space $(\mathcal{X}, d)$ is open if for each $x \in E$ there exists some $\epsilon_x > 0$ such that the entire ball

$$B_{\epsilon_x}(x) = \{\xi \in \mathcal{X} : d(x, \xi) < \epsilon_x\} \subset E$$

lies within $E$. A set $F \subset \mathcal{X}$ is closed if its complement $F^c = \{x \in \mathcal{X} : x \notin F\}$ is open.

a. Prove that $(0, 1)$ is open in $\mathcal{X} = \mathbb{R}$.

b. Prove that any union $U = \bigcup E_\alpha$ of open sets is also open.

c. Show by example that the union $U = \bigcup F_\alpha$ of closed sets may not be closed.
Problem 3: Recall that a set $K$ in a metric space $(X, d)$ is compact\(^2\) if every open cover $K \subset \bigcup_{\alpha} U_{\alpha}$ admits a finite sub-cover $K \subset \bigcup_{i=1}^{p} U_{\alpha_i}$, and that a function $f(\cdot) : X \to Y$ from one metric space to another is continuous if for every open set $U \subset Y$, $f^{-1}(U) := \{x : f(x) \in U\}$ is an open set in $X$.

a. Prove that every compact set $K$ is also closed.

b. If $K$ is a compact set and $F \subset K$ is a closed subset, prove that $F$ is also compact.

c. If $f : X \to \mathbb{R}$ is a continuous real-valued function and $K \subset X$ is compact, prove that the supremum

$$M := \sup_{x \in K} f(x)$$

is finite.

d. Show\(^3\) this can fail if $f$ is not continuous—i.e., give an example of an unbounded (but finite) function $f$ on a compact set $K$.

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\(^2\)The Heine-Borel theorem says in Euclidean space any closed & bounded set is compact, but that doesn’t hold in general. For example, the unit ball $B := \{f : \int_{0}^{1} |f(x)|^2 \, dx \leq 1\}$ is closed and bounded in $L_2((0,1])$ but is not compact, since the sequence of functions $\{f_n(x) := \sqrt{2} \sin(n\pi x)\} \subset B$ has no limit point in $B$.

\(^3\)Suggestion: take $K = [0, 1]$ on $X = \mathbb{R}$, and define $f(x)$ by cases. What cases?
Problem 4:

a. Let $K_\alpha$ be compact for each index $\alpha$ and suppose that each finite intersection $\bigcap_{j=1}^{n} K_{\alpha_j} \neq \emptyset$ is non-empty. Prove that $\bigcap_{\alpha} K_\alpha \neq \emptyset$.

b. If $f : \mathcal{X} \to \mathbb{R}$ is real-valued and continuous with supremum $M := \sup_{x \in K} f(x)$ on a compact set $K \subset \mathcal{X}$, prove that there exists some $x^* \in K$ for which $f(x^*) = M$—i.e., that the supremum is attained.
Problem 5:

a. Give an example of a closed set $C \subset \mathbb{R}$ that is not compact.

b. Give an example of a set $A \subset \mathbb{R}$ that is neither closed nor open.

c. Give an example of a set $B \subset \mathbb{R}$ that is both closed and open.
Problem 6: Evaluate the sums and integrals below for every value of \( p \in \mathbb{R} \) (some expressions might be infinite or undefined for some values of \( p \)):

a. \( \int_{0}^{1} x^p \, dx = \)

b. \( \int_{1}^{\infty} x^p \, dx = \)

c. \( \int_{0}^{\infty} e^{-px} \, dx = \)

d. \( \sum_{n=2}^{9} p^n = \)

e. \( \sum_{n=1}^{\infty} p^n = \)

f. \( \sum_{n=7}^{\infty} n \, p^n = \)

g. \( \int_{0}^{\infty} x \, e^{-px^2} \, dx = \)

h. \( \int_{0}^{x} \sin(\ln u) \, du = \)

i. \( \int_{0}^{\pi} e^{-p\cos(x)} \sin(x) \, dx = \)
Problem 7: Which of the following sums and integrals converges (to a finite limit)? Why? You need not evaluate the limit.

a. T F \( \int_{2}^{\infty} \frac{\ln(e^x - 2)}{x^3 + 1} \, dx \) converges:

b. T F \( \sum_{n=0}^{\infty} \frac{3^n(n!)^2}{(2n)!} \) converges:

c. T F \( \sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^{3/2}} \) converges:

d. T F \( \int_{0}^{\infty} \frac{\sin x}{x^{3/2}} \, dx \) converges:

e. T F \( \int_{0}^{\infty} \frac{dx}{\sqrt{x + x^2}} \) converges:

f. T F \( \int_{0}^{1} \frac{\tan x}{x^2} \, dx \) converges: