Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2018 Nov 14, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, Simplify.

Good luck!

Print Name Clearly: __________________________
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<td>Total:</td>
<td>/100</td>
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Problem 1: Let $\Omega := (0, \infty)$ with the Borel sets for $\mathcal{F}$ and probability measure given by

$$P\{ (a, b] \} = e^{-a} - e^{-b}, \quad 0 < a < b < \infty.$$ 

Let $X_n(\omega) := \omega^n / n!$ and $Y_n := X_n / 2^n$ on $\Omega$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots \}$, with partial sums $S_n := \sum_{j=0}^{n} X_j$ and $T_n := \sum_{j=0}^{n} Y_j$.

a) (2) Find the limits. Simplify!

$$S := \lim_{n \to \infty} S_n = \quad \quad T := \lim_{n \to \infty} T_n =$$

b) (2) Show that $E[X_n] = 1$ for all $n \in \mathbb{N}_0$.

c) (4) For all $n \in \mathbb{N}_0$ and $p \geq 1$, find

$$\| X_n \|_p =$$
Problem 1 (cont’d): Still $P(d\omega) = e^{-\omega} d\omega$ on $\Omega = \mathbb{R}_+$, $X_n(\omega) := \omega^n/n!$, $Y_n := X_n/2^n$, $S_n := \sum_{j=0}^{n} X_j$ and $T_n := \sum_{j=0}^{n} Y_j$.

d) (4) Does the Monotone Convergence Theorem apply to $\{S_n\}$? ☐ Yes ☐ No. If so, what does it say? If not, why not?

e) (4) Does the Dominated Convergence Theorem apply to $\{S_n\}$? ☐ Yes ☐ No. If so, what does it say? What’s the dominator? If not, why not?

f) (4) Does the Dominated Convergence Theorem apply to $\{T_n\}$? ☐ Yes ☐ No. If so, what does it say? What’s the dominator? If not, why not?
Problem 2: Let \( \{X_n\} \) be iid with CDF
\[
F(x) = P[X_n \leq x] = 1 - (1 + x)^{-2}, \quad x > 0
\]
and set \( S_n := \sum_{j=1}^{n} X_j \) and \( M_n := \min_{1 \leq j \leq n} X_j \).

a) (2) Find the probability density function \( f(x) \) for \( X_n \).

b) (4) Fix \( n \in \mathbb{N} \). For which \( p > 0 \) is \( X_n \in L_p \)?

c) (5) Does \( S_n/n \) converge as \( n \to \infty \)? \( \bigcirc \) Yes \( \bigcirc \) No

If so, to what limit, in what sense\(^1\), and why? If not, why not?

d) (5) Fix \( n \in \mathbb{N} \). For what \( p > 0 \) is \( M_n \in L_p \)?

e) (4) Does \( M_n \) converge as \( n \to \infty \)? \( \bigcirc \) Yes \( \bigcirc \) No

If so, to what limit, in what sense\(^1\), and why? If not, why not?

\(^1\)In case it converges in more than one sense, give any correct sense with a matching answer to “why?”.
Problem 3: Still \( \{X_n\} \) are IID with CDF \( F(x) = 1-(1+x)^{-2} \) for \( x > 0 \).

a) (4) Does the Central Limit Theorem apply to \( \{X_n\} \)? ⃝ Yes ⃝ No
If so, what does it say? If not, why not?

b) (4) Let \( Y_n := \mathbf{1}_{\{X_n > 1\}} \) and \( T_n := \sum_{j=1}^{n} Y_j \).
Find the mean and variance of \( Y_n \) and \( T_n \):

\[
\begin{align*}
E[Y_n] &= \underline{\text{_______}} & E[T_n] &= \underline{\text{_______}} \\
V[Y_n] &= \underline{\text{_______}} & V[T_n] &= \underline{\text{_______}}
\end{align*}
\]
Problem 3 (cont’d): Still \( \{X_n\} \) are IID with CDF \( F(x) = 1 - (1 + x)^{-2} \) for \( x > 0 \) and \( T_n := \sum_{j=1}^{n} Y_j \) with \( Y_j := 1_{\{X_j > 1\}} \).

c) (6) Find the ch.f. \( \phi_n(\omega) := \mathbb{E} \exp(i\omega Z_n) \) for \( Z_n := (T_n - n/4)/\sqrt{n} \):
\[ \phi_n(\omega) = \]

d) (6) For large \( n \) this has to be approximately \( \phi_n(\omega) \approx \exp(-\kappa \omega^2/2) \) for some \( \kappa > 0 \). What theorem says so? And what is \( \kappa \)? \( \kappa = \)
Problem 4: On \((\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)\), let \(\mathcal{F}_n = \sigma\{(0, j/2^n] : 1 \leq j \leq 2^n\}\) and let \(X(\omega) := 1/\omega, Y(\omega) := \omega^2, Z(\omega) := 1_{((0,3/8]])}\).

a) (4) Is \(E[X \mid \mathcal{F}_2]\) well-defined? ○ Yes ○ No
If so, give its value at \(\omega = 1/3\); if not, say why.
\(E[X \mid \mathcal{F}_2](1/3) = \)

b) (4) Is \(E[Y \mid \mathcal{F}_2]\) well-defined? ○ Yes ○ No
If so, give its value at \(\omega = 1/3\); if not, say why.
\(E[Y \mid \mathcal{F}_2](1/3) = \)

c) (4) Is \(E[Z \mid \mathcal{F}_2]\) well-defined? ○ Yes ○ No
If so, give its value at \(\omega = 1/3\); if not, say why.
\(E[Z \mid \mathcal{F}_2](1/3) = \)

d) (4) Find the indicated conditional expectation:
\(E[Y \mid Z] = \)

e) (4) Find the indicated conditional expectation:
\(E[Z \mid Y] = \)
Problem 5: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space \((\Omega, \mathcal{F}, P)\).

a) T F If RVs \(X_n\) decrease to a limit \(X \in L_1\), and if each \(X_n \leq Z\) for some \(Z \in L_1\), then \(X_n \to X\) in \(L_1\).

b) T F If \(\{X_j\}\) are independent w/ch.f.s \(\phi_j(\omega) := \mathbb{E}[e^{i\omega X_j}]\), then \(\sum_{j=1}^n X_j\) has ch.f. \(\phi(\omega) := \prod_{j=1}^n \phi_j(\omega)\).

c) T F If \(\{X, Y, Z\}\) are iid and \(P[X > 0] = 1\) then \(\mathbb{E}[X/(Y+Z)] = 1/2\).

d) T F If \(X\) and \(Y\) are independent and \(Y\) has pdf \(f(y)\) then \(Z := X + Y\) has an absolutely-continuous distribution too with some pdf \(g(z)\).

e) T F If \(\{X_n\}\) satisfy \(P[|X_n| \leq n] = 1\) and converge a.s. to a limit \(X\), then \(X_n \to X\) in \(L_1\).

f) T F If \(\mathbb{E}\sqrt{|X_n|} \to 0\) then also \(\mathbb{E}|X_n|^2 \to 0\).

g) T F If \(P[A] \leq 1/4\) then \(\|X1_A\|_1 \leq \frac{1}{2}\|X\|_2\) for any RV \(X\).

h) T F If \(G \subset \mathcal{F}, Y := \mathbb{E}[X \mid G]\), and \(0 \leq X \in L_2\), then \(\|X\|_2 \geq \|Y\|_2\).

i) T F If \(X\) has ch.f. \(\phi(\omega)\) then \(Y := \sqrt{X}\) has ch.f. \(\phi(\omega/2)\).

j) T F If \(X_n \to X\) in \(L_2\) then, for some \(n_k \to \infty\), \(X_{n_k} \to X\) in \(L_4\).
Blank Worksheet
Another Blank Worksheet
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>Be($\alpha$, $\beta$)</td>
<td>$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha + \beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$</td>
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<tr>
<td>Binomial</td>
<td>Bi($n$, $p$)</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>Ex($\lambda$)</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>Ga($\alpha$, $\lambda$)</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>Ge($p$)</td>
<td>$f(x) = p q^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
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<tr>
<td></td>
<td></td>
<td>$f(y) = p q^{y-1}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
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<tr>
<td>HyperGeo.</td>
<td>HG($n$, $A$, $B$)</td>
<td>$f(x) = \frac{(\lambda)^n (\mu_x)}{\binom{\alpha}{n} \mu_{x}}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$n P$</td>
<td>$n P \left(1-P\right)\frac{N-n}{N-1}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>Lo($\mu$, $\beta$)</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1 + e^{-(x-\mu)/\beta}]^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2/3$</td>
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<tr>
<td>Log Normal</td>
<td>LN($\mu$, $\sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\log x - \mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\mu + \sigma^2/2$</td>
<td>$e^{2\mu + \sigma^2/2}$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>NB($\alpha$, $p$)</td>
<td>$f(x) = \binom{\alpha - 1}{x} p^\alpha q^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
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<tr>
<td></td>
<td></td>
<td>$f(y) = \binom{\alpha - 1}{\alpha - y} p^\alpha q^{y-\alpha}$</td>
<td>$y \in {\alpha, \ldots}$</td>
<td>$\alpha/p$</td>
<td>$\alpha q/p^2$</td>
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<td>Normal</td>
<td>No($\mu$, $\sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
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<tr>
<td>Pareto</td>
<td>Pa($\alpha$, $\epsilon$)</td>
<td>$f(x) = \left(\frac{\alpha}{\epsilon}\right) (1 + x/\epsilon)^{-\alpha-1}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\epsilon}{\alpha - 1}$ if $\alpha &gt; 1$</td>
<td>$\frac{\epsilon^2\alpha}{(\alpha - 1)^2(\alpha - 2)}$ if $\alpha &gt; 2$</td>
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<tr>
<td></td>
<td></td>
<td>$f(y) = \alpha \epsilon^{\alpha y}/y^{\alpha+1}$</td>
<td>$y \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon\alpha}{\alpha - 1}$ if $\alpha &gt; 1$</td>
<td>$\frac{\epsilon^2\alpha}{(\alpha - 1)^2(\alpha - 2)}$ if $\alpha &gt; 2$</td>
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<tr>
<td>Poisson</td>
<td>Po($\lambda$)</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma(\nu_1/2 + \nu_2/2) \Gamma(\nu_1/2)}{\Gamma(\nu_1/2) \Gamma(\nu_2/2)} \times (x / 2) \frac{\nu_2}{\nu_2 - 2}$ if $\nu_2 &gt; 2$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2 - 2}$ if $\nu_2 &gt; 2$</td>
<td>$\left(\frac{\nu_2}{\nu_2 - 2}\right)^2 \frac{2(\nu_1 + \nu_2 - 2)}{(\nu_1 + \nu_2 - 4)}$ if $\nu_2 &gt; 4$</td>
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<tr>
<td>Student</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma(\nu/2 + 1/2)}{\Gamma(\nu/2)\sqrt{\nu/\pi}} \left[1 + x^2/\nu\right]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$ if $\nu &gt; 1$</td>
<td>$\frac{\nu}{\nu - 2}$ if $\nu &gt; 2$</td>
</tr>
<tr>
<td>Uniform</td>
<td>Un($a$, $b$)</td>
<td>$f(x) = \frac{1}{b - a}$</td>
<td>$x \in \langle a, b \rangle$</td>
<td>$\frac{a + b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>We($\alpha$, $\beta$)</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\Gamma(1/\beta^{1/\alpha})$</td>
<td>$\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)$</td>
</tr>
</tbody>
</table>