Final Examination

STA 711: Probability & Measure Theory

Saturday, 2019 Dec 14, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible simplify.

Good luck.

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<tr>
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Total: /160

Print Name: ________________________________
Problem 1: A few questions about $L_p(\Omega, \mathcal{F}, P)$ and convergence:

a) (5) Suppose that the sequence $\{X_n\}$ converges both a.s. and in $L_2$—say, $X_n \to Y_1$ (a.s.), and $X_n \to Y_2$ in $L_2$. Prove $P[Y_1 = Y_2] = 1$.

b) (5) Suppose $X_1, X_2, X_3$ are all RVs on $(\Omega, \mathcal{F}, P)$ with

$$\|X_1\|_1 = 1 \quad \|X_2\|_2 = 2 \quad \|X_3\|_3 = 3$$

Set $S := X_1 + X_2 + X_3$. For what $p > 0$ is $S \in L_p$?

Give and justify a bound for $\|S\|_p$:

$$\|S\|_p \leq \ldots$$
Problem 1 (cont’d): More about $L_p(\Omega, \mathcal{F}, P)$

c) (5) Fix $p > 1$. Let $A \in \mathcal{F}$ be an event with probability $a := \mathbb{P}[A]$ and let $X \in L_p(\Omega, \mathcal{F}, P)$ be a positive RV with norm $x := \|X\|_p$. Find and justify a non-trivial upper bound (one that tends to zero as $a \to 0$ or $x \to 0$):

$\mathbb{E}[|X\mathbf{1}_A|] \leq \ldots$

d) (5) Let $\{X, Y\} \subset L_1$ be independent and non-negative, but not in $L_2$. Prove that $XY \in L_1$. Suggestion: For $t > 0$, consider $(X \wedge t)(Y \wedge t)$, where $x \wedge y := \min(x, y)$ for $x, y \in \mathbb{R}$. Then what?
Problem 2: Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel sets on $\mathbb{R}$. Define two probability measures on $(\Omega, \mathcal{F})$ by

$$
P(d\omega) := \frac{1}{2} e^{-|\omega|} d\omega \quad \quad Q(d\omega) := 1_{\{\omega > 0\}} e^{-\omega} d\omega.
$$

Set $\mathcal{B} := \{(-a, a) : a \in \mathbb{R}_+\}$ and $\mathcal{G} := \sigma(\mathcal{B})$.

a) (5) Prove that $P$ and $Q$ agree on all of $\mathcal{G}$.

b) (5) Prove that $P$ and $Q$ do not agree on all of $\mathcal{F}$.
Problem 2 (cont’d): Still $P(d\omega) := \frac{1}{2} e^{-|\omega|} \, d\omega$ and $Q(d\omega) := 1_{\{\omega > 0\}} e^{-\omega} \, d\omega$, with $\mathcal{B} := \{(a, a) : a \in \mathbb{R}_+\}$ and $\mathcal{G} := \sigma(\mathcal{B})$.

c) (5) Give an example of a non-constant $\mathcal{G}$-measurable RV $X$.
$X(\omega) =$

d) (5) Let $\{X_n\}$ be iid RVs with distribution $P$. Find a non-trivial DF $G$ such that the maximum $X^*_n := \max\{X_j : 1 \leq j \leq n\}$ satisfies

$$(\forall x \in \mathbb{R}) \quad P\left[ (X^*_n - \log n) \leq x \right] \to G(x)$$
Problem 3: Let \( \{A_n\} \subset \mathcal{F} \) and \( \{X_n\} \subset L_1(\Omega, \mathcal{F}, P) \) with \( \|X_n\|_1 \leq 1 \) and \( P(A_n) \to 0 \).

a) (8) Does it follow that \( \mathbb{E}X_n 1_{A_n} \to 0 \)? \( \bigcirc \) Yes \( \bigcirc \) No. Prove it, or give a counter-example:

b) (8) Does it follow that \( \mathbb{E}X_1 1_{A_n} \to 0 \)? \( \bigcirc \) Yes \( \bigcirc \) No. Prove it, or give a counter-example:

c) (4) Would either of your answers to a) or b) change if we have \( \{X_n\} \subset L_4(\Omega, \mathcal{F}, P) \) with \( \|X_n\|_4 \leq 42 \)? Explain.
Problem 4: Let \( \{A_n\} \) be independent events on some probability space \((\Omega, \mathcal{F}, P)\) with \( P(A_n) = 2^{-2n} = 4^{-n} \) and for \( n \geq 0 \) set
\[
X_n := 2^n 1_{A_n} \quad Y := \sum_{n=0}^{\infty} X_n \quad Z_n := \prod_{0 \leq j < n} X_j
\]

a) (4) Show that \( Y \) is finite almost-surely:

b) (4) For which \( 0 < p \leq \infty \) and \( n \in \mathbb{N} \) is \( X_n \in L_p \)? Why?

c) (4) For which \( 0 < p \leq \infty \) is \( Y \in L_p \)? Why?

d) (4) Is the sequence \( \{Z_n^p\} \) a Martingale for some \( p > 0 \)? \( \bigcirc \) Yes \( \bigcirc \) No Why?

e) (4) In what sense(s) does \( Z_n \) converge, and to what limit?
\( \bigcirc \) a.s. \( \bigcirc \) pr. \( \bigcirc \) \( L_1 \) \( \bigcirc \) \( L_2 \) \( \bigcirc \) \( L_\infty \) Limit:
Problem 5: Let $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Ex}(\theta)$ be independent.

a) (4) Is the distribution $\mu(dz)$ of $Z := X \wedge Y$: $\bigcirc$ Absolutely Continuous, $\bigcirc$ Discrete, or $\bigcirc$ Neither?¹ For each $z \in \mathbb{R}$, find:

$P[Z = z] = \mu(\{z\}) = $

b) (4) Is the distribution $\nu(ds)$ of $S := X + Y$: $\bigcirc$ Absolutely Continuous, $\bigcirc$ Discrete, or $\bigcirc$ Neither? For each $s \in \mathbb{R}$, find:

$P[S = s] = \nu(\{s\}) = $

c) (4) For fixed $\theta > 0$, find $\lambda \in \mathbb{R}_+$ to achieve $P[X < Y] = \frac{1}{2}$:

$\lambda = $

¹Recall $x \wedge y := \min(x, y)$ for $x, y \in \mathbb{R}$
Problem 5 (cont’d): Still $X \sim \text{Po}(\lambda)$, $Y \sim \text{Ex}(\theta)$, and $X \perp \perp Y$.

d) (4) Find the MGF $M(t) := E[\exp(tS)]$ for the sum $S := X + Y$. For which $t \in \mathbb{R}$ is $M(t) < \infty$ finite?

$M(t) =$

e) (4) Find the ch.f. $\chi(\omega) := E[\exp(i\omega S)]$ for the sum $S := X + Y$. For which $\omega \in \mathbb{R}$ is $|\chi(\omega)| < \infty$ finite?

$\chi(\omega) =$
Problem 6: The random variables $X$ and $Z$ are independent, with distributions

$$X \sim \text{No}(0, 1) \quad P[Z = +1] = 1/2 = P[Z = -1]$$

and product $Y := XZ$. Simplify all answers.

a) (6) What is the probability distribution of $Y$?

b) (4) What is the covariance of $X$ and $Y$?

c) (5) Are $X$ and $Y$ independent?  \quad \bigcirc \text{Yes} \quad \bigcirc \text{No} \quad \text{Why?}

d) (5) Are $Y$ and $Z$ independent?  \quad \bigcirc \text{Yes} \quad \bigcirc \text{No} \quad \text{Why?}
Problem 7: Let $X \sim \text{Ex}(a)$, $Y \sim \text{Ex}(b)$, and $Z \sim \text{Ex}(c)$ be independent RVs on $(\Omega, \mathcal{F}, P)$ (see p. 13 for the pdf, mean, etc. of $\text{Ex}(\lambda)$). Find (and simplify):

a) (5) $P[Y < Z] = \int_0^\infty \int_y^\infty b c e^{-by-cz} \, dz \, dy = \ldots$

b) (5) $P[X < Y < Z] = \ldots$

c) (5) $P[X < Y \mid Y < Z] = \ldots$

d) (5) $E[XY/Z] = \ldots$
Problem 8: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, P)$.

a) T F For any real number $x \in \mathbb{R}, x \leq e^x - 1$.

b) T F If events $A, B, C$ are independent then $(A \cup B)$ and $C$ are independent too.

c) T F If events $A, B, C$ are independent then $(A \cap B)$ and $C$ are independent too.

d) T F If $\{X_i\}$ are iid and $P[X_i \in (0, 1)] = 1$ then $\prod_{j=1}^{n} X_j \to 0$ a.s.

e) T F If the distribution $\mu(B) := P[X \in B]$ of $X$ is neither absolutely-continuous nor discrete, then it must be singular continuous.

f) T F If $E|Z| = \infty$ then $E[Z \mid \mathcal{G}]$ is not defined.

g) T F If $X \sim \text{Ex}(\lambda)$ then, for $x > 0$, $P[X = x] = \lambda e^{-\lambda x}$.

h) T F If $E[e^{i\omega X}] = \cos(\omega)$ then $X \sim \text{Un}((-\pi, \pi])$.

i) T F If $\sigma(X) = \sigma(Y)$ then $X = \phi(Y)$ for some Borel function $\phi$.

j) T F If $\{M_n\}$ is a martingale then $\zeta_n := (M_n - M_{n-1})$ are independent $L_1$ RVs with mean zero.

Fall 2019 11 Sat Dec 14: 2 \to 5 pm
Blank Worksheet
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
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<tbody>
<tr>
<td>Beta</td>
<td>$Be(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{x}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$Bi(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$npq$ if $q = 1 - p$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$Ex(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$Ga(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda \alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
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<tr>
<td>Geometric</td>
<td>$Ge(p)$</td>
<td>$f(x) = p q x^p$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$ if $q = 1 - p$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$HG(n, A, B)$</td>
<td>$f(x) = \binom{A+B}{x} \binom{A}{x} / \binom{A+B}{A}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$nP$</td>
<td>$nP(1-P) \frac{N-n}{N-1}$ if $P = \frac{A}{A+B}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$Lo(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta [1-e^{-(x-\mu)/\beta}]^{\beta+1}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 / 3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$LN(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{x \sqrt{2\pi} \sigma^2} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2}(e^{\sigma^2-1})$</td>
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<tr>
<td>Neg. Binom.</td>
<td>$NB(\alpha, p)$</td>
<td>$f(x) = \frac{(p+\alpha-1)<em>p}{(\alpha-x)</em>\alpha} p^x q^{\alpha-x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$ if $q = 1 - p$</td>
</tr>
<tr>
<td>Normal</td>
<td>$No(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
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<tr>
<td>Pareto</td>
<td>$Pa(\alpha, \epsilon)$</td>
<td>$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{x}{\alpha-1}$ if $\alpha &gt; 1$</td>
<td>$\frac{x^2}{(\alpha-2)(\alpha-1)}$ if $\alpha &gt; 2$</td>
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<tr>
<td>Poisson</td>
<td>$Po(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor $F$</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2}) (\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[ 1 + \frac{\nu_2}{\nu_2-2} x \right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 &gt; 2$</td>
<td>$\left( \frac{\nu_2}{\nu_2-2} \right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2 &gt; 4$</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu \pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\nu/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$ if $\nu &gt; 1$</td>
<td>$\frac{\nu}{\nu-2}$ if $\nu &gt; 2$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$Un(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
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<tr>
<td>Weibull</td>
<td>$We(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td></td>
<td>$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}} \frac{\Gamma(1+2/\alpha-1)}{\beta^{2/\alpha}}$ if $\beta &gt; 1$</td>
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