Sta 711: Homework 1

Fields and $\sigma$-fields

1. Enumerate the class $\mathcal{K}$ of all $\sigma$-fields $\mathcal{F}$ on the three-point set $\Omega = \{a, b, c\}$ that contain the singleton $\{a\}$, i.e., that satisfy $C \subset \mathcal{F}$ for $C := \{\{a\}\}$. What is $\sigma(C)$?

2. Prove that for any two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on any set $\Omega$, the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a field.

3. Find a set $\Omega$ and two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega$ for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a field.

4. Suppose a collection $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of $\sigma$-fields on a set $\Omega$ satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\bigcup \mathcal{F}_j$ is a field? (the answer is “yes” — show why)

5. Under the same conditions, must $\bigcup \mathcal{F}_j$ be a $\sigma$-field? (this one is “no” — find a counterexample. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\bigcup A_n \notin \mathcal{F}_j$ for every $j$, hence $\bigcup A_n \notin \bigcup \mathcal{F}_j$).
Dyadic Rational Probability Spaces

For problems 6–9, let \( \Omega = \mathbb{Q}_2 := \{j/2^n : j \in \{1, 2, \cdots , 2^n \}, n \in \mathbb{N} \} \) be the dyadic rational numbers in the half-open unit interval, and let

\[
C = \{(0, b] \cap \mathbb{Q}_2 : b \in \mathbb{Q}_2, \ 0 < b \leq 1 \}
\]

(1)
denote the collection of half-open intervals of dyadic rationals \( (0, b] = \{q \in \mathbb{Q}_2 : 0 < q \leq b \} \) with left endpoint zero. Every \( \Omega \) on this page contains only dyadic rational numbers.

Recall that a real-valued set function \( P \) on a \( \sigma \)-algebra \( G \) of subsets of a space \( \Omega \) is a “probability measure” (PM) if and only if it satisfies the three rules:

- (\( \forall A \in G \)) \( P(A) \geq 0 \);
- (\( \forall \{A_i\} \subset G, \ A_i \cap A_j = \emptyset \) \( P(\bigcup A_i) = \sum P(A_i) \);
- \( P(\Omega) = 1 \).

6. Let \( n \in \mathbb{N} \) be a FIXED positive integer (like three) and set

\[
B_n := \{(0, j/2^n], j \in \{0, 1, \cdots , 2^n \}\},
\]

the collection of half-open intervals in \( \Omega \) of dyadic rationals from zero up to an integral multiple of \( 2^{-n} \). Describe the elements of the \( \sigma \)-field

\[
F_n := \sigma(B_n)
\]
generated by \( B_n \), for fixed \( n \in \mathbb{N} \). How many elements does \( B_n \) have? How many distinct elements does \( F_n \) have? What are they? Suggestion: Try \( B_0, B_1 \) and \( B_2 \) first, by hand. Is there a partition that generates \( F_n \)?

7. What is the field \( F_0 := \mathcal{F}(C) \) of subsets of \( \mathbb{Q}_2 \) generated by the class \( C \) of Eqn (1)? (hint: Do problems (4) and (6) first). Try to describe it in just a few words, without using any symbols besides \( \mathbb{Q}_2 \). Don’t just echo the definition!

8. Describe simply and clearly in no more than five words or symbols (seriously, three should be enough) the \( \sigma \)-field \( \mathcal{F} := \sigma(C) \) of subsets of \( \mathbb{Q}_2 \) generated by \( C \). Don’t just echo the definition!

9. Define a set function \( \lambda_0 \) on \( C \) by

\[
\lambda_0\left( (0, b] \right) = b
\]

Show that there does not exist a probability measure \( \lambda \) on \( (\mathbb{Q}_2, \mathcal{F}) \) that extends \( \lambda_0 \), i.e., one for which \( \lambda( (0, b] ) = b \) for all \( b \in \mathbb{Q}_2 \) (Hint: Exactly what does the function \( F(x) := \lambda\left( (0, x] \right), 0 \leq x \leq 1 \) look like near \( x \in \mathbb{Q}_2 \), for any PM \( \lambda \) on \( \mathbb{Q}_2 \)?)

Last edited: September 16, 2019