Convergence

1. Let $X$ be a strictly positive random variable. Show that:
   
   (a) $\lim_{n \to \infty} n E(\frac{1}{X} 1_{X>n}) = 0$.
   
   (b) $\lim_{n \to \infty} n^{-1} E(\frac{1}{X} 1_{X>n-1}) = 0$.

2. Let $X \sim \text{Un}(0, 4]$ be uniformly distributed on the interval $(0, 4]$, and set $Y := \frac{1}{X}$ and $Z := \log(4Y)$. Suggestion: First find out what is the distribution of $Z$, by computing $P[Z > z]$ for $z \in \mathbb{R}$. Use $\varphi(x) := |x|$ for the Markov inequality questions.
   
   (a) What bound does Markov’s inequality give for $P[X > 3]$?
   
   (b) What bound does Chebychev’s inequality give for $P[|X - 2| > 1]$?
   
   (c) What bound does Markov’s inequality give for $P[Y > 1]$?
   
   (d) What bound does Markov’s inequality give for $P[Z > 2]$?
   
   (e) What are the exact values of $P[X > 3]$, $P[|X - 2| > 1]$, $P[Y > 1]$, and $P[Z > 2]$?

3. Let $A$ and $B$ be events in $(\Omega, \mathcal{F}, P)$ with probabilities $a = P(A)$ and $b = P(B)$ respectively. Show that $P(A \cap B) \leq \sqrt{ab}$.

4. Suppose $\{X_n\}, X$ are real valued RVs defined on a probability space $(\Omega, \mathcal{F}, P)$ and that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$. Show that for every $\epsilon > 0$, there is an event $\Lambda_\epsilon$ with $P(\Lambda_\epsilon) < \epsilon$ and
   
   $\sup_{\omega \in \Lambda_\epsilon} |X(\omega) - X_n(\omega)| \to 0$ as $n \to \infty$.

   Thus the convergence is uniform except on an arbitrarily small set. (For more on this result, called Egorov’s Theorem, see page 89 of the text.)

5. For a random variable $X$, $1 < p < q < \infty$, show\(^1\) that
   
   \[ 0 \leq \|X\|_1 \leq \|X\|_p \leq \|X\|_q \leq \|X\|_\infty \]

6. For $1 < p < q < \infty$, show that
   
   $L_\infty \subset L_q \subset L_p \subset L_1$

   where $L_p := \{X : \|X\|_p < \infty\}$.

\(^1\)Hint: Jensen’s inequality may help for some parts
7. The “Moment Generating Function” (MGF) of a real-valued random variable $X$ (or of its distribution $\mu(dx)$) is the extended real-valued function $M_X(t) := \mathbb{E}\exp(tX) = \int_{\mathbb{R}} e^{tx} \mu(dx)$ of $t \in \mathbb{R}$. Show that a nonnegative random variable $X \geq 0$ is in $L_1$ if $M_X(t) < \infty$ for any $t > 0$. Show that the converse may fail—i.e., there exist $X \geq 0$ in $L_1$ for which $M_X(t) = \infty$ for all $t > 0$.

8. Show that Minkowski’s Inequality fails for $0 < p < 1$—i.e., find $(\Omega, \mathcal{F}, \mathbb{P})$ and $X, Y \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for which $\|X + Y\|_p > \|X\|_p + \|Y\|_p$ for some $0 < p < 1$. 