Final Examination

STA 711: Probability & Measure Theory

Monday, 2018 Dec 17, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, please ask me to clarify it. Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in closed form with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible simplify.

Good luck.

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Print Name: ________________________________
**Problem 1:** Let \( \{A_n\} \subset \mathcal{F} \) be independent events with probabilities \( P[A_n] = 1/n \), and let \( X_n := 1_{A_n} \) be their indicator RVs.

a) (5) Does \( \sum_n X_n \) converge a.s. to an \( \mathbb{R} \)-valued limit \( X \)? ○ Yes ○ No Why?

b) (5) Does \( \sum_n X_n^2 \) converge a.s. to an \( \mathbb{R} \)-valued limit \( X \)? ○ Yes ○ No Why?

c) (5) Does \( \sum_n X_n^2 \) converge in \( L_1 \) to an \( \mathbb{R} \)-valued limit \( X \)? ○ Yes ○ No Why?

d) (5) Does \( \sum_n n X_{2^n} \) converge in \( L_p \) to an \( \mathbb{R} \)-valued limit \( X \) for each \( 0 < p < \infty \)? ○ Yes ○ No Why?
Problem 2: Let \( \{X_n\} \) and \( Y \) be real-valued random variables on \((\Omega, \mathcal{F}, P)\) such that \( X_n \to Y \) a.s. For each \( n \in \mathbb{N} \), \( \mathbb{E}[X_n^2] \leq 100 \).

a) (5) Does it follow that \( Y \in L^2 \)? ○ Yes ○ No Why?

b) (5) Does it follow that \( X_n \to Y \) in \( L^2 \)? ○ Yes ○ No Proof or counter-example:

c) (5) Is \( \mathbb{P}[|X_n - Y| > \epsilon] \) summable for each \( \epsilon > 0 \)? ○ Yes ○ No Proof or counter-example:

d) (5) Is \( \mathbb{P}[|X_1 - Y|^2 > n\epsilon] \) summable for each \( \epsilon > 0 \)? ○ Yes ○ No Proof or counter-example:
Problem 3: Let $X \sim \text{Ex}(\lambda)$ and $Y \sim \text{Ge}(p)$ be independent, with pdf $f(x) = \lambda e^{-\lambda x}1_{\{x>0\}}$ and pmf $p(y) = pq^y, \ y \in \mathbb{N}_0$, respectively, where $q := 1 - p$.

a) (5) Find $P[Y > X] =$

b) (5) Is the distribution $\mu(dz)$ of $Z := X + Y$ Absolutely Continuous, Discrete, or Neither? Give its survival function at all $z \in \mathbb{R}$. $\bar{F}(z) := P[Z > z] =$

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Problem 3 (cont’d): Still $X \sim \text{Ex}(\lambda) \perp Y \sim \text{Ge}(p)$ and $Z := X + Y$.

c) (6) Find the characteristic functions of all three RVs:

\[
\chi_X(\omega) =
\]

\[
\chi_Y(\omega) =
\]

\[
\chi_Z(\omega) =
\]

d) (4) Find the indicated conditional expectation:

\[
E[Z \mid X] =
\]
Problem 4: Let $Z \sim \text{No}(0,1)$ and set $X := (Z \lor 0)$, the maximum of $Z$ and zero.

a) (5) Is the distribution $\mu(dx)$ of $X := (Z \lor 0)$ Absolutely Continuous, Discrete, or Neither? Give its survival function at all $x \in \mathbb{R}$, or some other representation of its distribution.

$b) (5)$ Find the moment generating function (MGF) of $X$. Your expression may include the normal CDF $\Phi(\cdot)$.

$c) (5)$ Find the mean of $X$ (use any method you like).

$d) (5)$ Every MGF satisfies $M(0) = 1$. Is there any other $t^* \neq 0$ for which this $M(t^*) = 1$? Why, or why not?
Problem 5: Let \( \{\xi_n\} \sim \text{Po}(n^2) \).

a) (5) Find the log ch.f.\(^1\) for \( X_n := \xi_n/n^2 \):
\[
\phi_n(\omega) = \log E[e^{i\omega X_n}] =
\]

b) (5) Show that \( \phi_n(\omega) \) converges as \( n \to \infty \), and find the limit \( \phi(\omega) \).
What distribution has ch.f. \( \exp(\phi(\omega)) \)?

---

\(^1\) Suggestion: First compute the ch.f. \( \phi(\theta) := E[e^{i\theta X}] \) for \( X \sim \text{Po}(\lambda) \).
Problem 5 (cont’d): Still \( \{\xi_n\} \sim \text{Po}(n^2) \).

c) (5) Find the log ch.f. for \( Y_n := (\xi_n/n) - n \):
\[
\psi_n(\omega) = \ldots
\]

d) (5) Show that \( \psi_n(\omega) \) converges as \( n \to \infty \), and find the limit \( \psi(\omega) \).
Identify the limiting distribution of \( \{Y_n\} \), which has ch.f. \( \exp(\psi(\omega)) \).
Problem 6: Let $X_0 := 1$ and, for $n \in \mathbb{N}$, let $X_n = 2X_{n-1}$ or $X_n = 0$ with probability $1/2$ each. Set $\tau := \inf\{n : X_n = 0\}$ and $\mathcal{F}_n := \sigma\{X_j : 1 \leq j \leq n\}$.

a) (6) Prove that $(X_n, \mathcal{F}_n)$ is a martingale (reminder: there are two conditions to verify).

b) (4) For each $p > 0$: is $\{X_n\}$ uniformly bounded in $L_p$? If so, by what?

c) (4) Does $\{X_n\}$ converge to some limit $X_\infty$ as $n \to \infty$? If so, to what limit, and in what sense(s)? If not, why not?

d) (4) Is $\tau$ in $L_1$? Prove it (and find $E[\tau]$) or disprove it.

e) (2) Find:

$E[X_\tau] =$

$E[X_{\tau \wedge 10}] =$

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Problem 7: Let $A, B, C$ be independent with probabilities $a, b, c$, respectively on $(\Omega, \mathcal{F}, P)$. Find:

a) (5) $P[A \cup B] =$

b) (5) $P[A \cup B \mid B \cup C] =$

c) (5) $P[A \cup B \cup C] =$

d) (5) $P[A \mid A \cup B \cup C] =$
Problem 8: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, P)$; $\phi, \psi$ are arbitrary Borel functions on $\mathbb{R}$.

a) T F If $X_n \to X$ a.s. then $\liminf_{n \to \infty} X_n = X$.

b) T F If $X = \phi(Z)$ and $Y = \psi(Z)$ then $X, Y$ can’t be independent.

c) T F If $g(\cdot)$ is continuous and $X_n \to X$ (pr.) then $g(X_n) \to g(X)$ (pr.).

d) T F If $X \perp \perp Y$ and $\phi, \psi$ are bounded functions $\mathbb{R} \to \mathbb{R}$ then $E[\exp (\phi(X) + \psi(Y))] = E[\exp (\phi(X))] \cdot E[\exp (\psi(Y))]$.

e) T F If $A, B \in \mathcal{F}$ then $\sigma\{A, B\} = \sigma\{1_A + 21_B\}$.

f) T F If $X \in L_1(\Omega, \mathcal{F}, P)$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ then 

$$E[E[X \mid \mathcal{H}] \mid \mathcal{G}] = E[X \mid \mathcal{G}]$$


g) T F If $\emptyset \neq \Lambda_1 \subset \subset \Lambda_2 \subset \subset \cdots \subset \subset \Lambda_n = \Omega$, then $\sigma\{\Lambda_j : 1 \leq j \leq n\}$ has $2^n$ elements.

h) T F If probability measures $P, Q$ agree on a field $\mathcal{G}_0$ then they agree on the $\sigma$-field $\mathcal{G} = \sigma(\mathcal{G}_0) \subset \mathcal{F}$ it generates.

i) T F If $0 \leq X \in L_1$ then $Y := \log(1 + X)$ satisfies $Y \in L_1$.

j) T F If each $X_j \in L_{p_j}$ for some $\{p_j\} \subset \mathbb{R}_+$ and if $\sum p_j < \infty$ then $X_+ := \sum X_j$ converges in $L_1$. 

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Blank Worksheet
Another Blank Worksheet
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>Be($\alpha$, $\beta$)</td>
<td>$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha + \beta}$</td>
<td>$\frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>Bi($n$, $p$)</td>
<td>$f(x) = \left(\begin{array}{c} n \end{array}\right) p^x q^{(n-x)}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>Ex($\lambda$)</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>Ga($\alpha$, $\lambda$)</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>Ge($p$)</td>
<td>$f(x) = pq^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>HG($n$, $A$, $B$)</td>
<td>$f(x) = \binom{\alpha}{n} \binom{\beta}{n} x^{\alpha n} y^{\beta n}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$np$</td>
<td>$np (1-P) \frac{N-p}{N-1}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>Lo($\mu$, $\beta$)</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta(1+e^{-(x-\mu)/\beta})^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 /3$</td>
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<tr>
<td>Log Normal</td>
<td>LN($\mu$, $\sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2} (e^{\sigma^2}-1)$</td>
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<tr>
<td>Neg. Binom.</td>
<td>NB($\alpha$, $p$)</td>
<td>$f(x) = \binom{\alpha}{x} p^x q^{\alpha-x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
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<tr>
<td>Normal</td>
<td>No($\mu$, $\sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>Pa($\alpha$, $\epsilon$)</td>
<td>$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\epsilon}{\alpha-1}$ if $\alpha &gt; 1$</td>
<td>$\frac{\epsilon^2}{(\alpha-1)(\alpha-2)}$ if $\alpha &gt; 2$</td>
</tr>
<tr>
<td>Poisson</td>
<td>Po($\lambda$)</td>
<td>$f(x) = \frac{\lambda e^{-\lambda}}{x!}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\frac{\lambda^x}{x!}$</td>
<td>$\frac{\lambda^x}{x!}$</td>
</tr>
<tr>
<td>Snedecor $F$</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\left(\frac{\nu_2}{\nu_2-2}\right)^{\frac{\nu_2}{2}}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)\left[1+\frac{\nu_2}{\nu_2-2}\right]^{\frac{\nu_1+\nu_2}{2}}} x^{\frac{\nu_1-2}{2}} \left[1+\frac{\nu_2}{\nu_2-2}\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 &gt; 2$</td>
<td>$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_1-4)}$ if $\nu_2 &gt; 4$</td>
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<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}\nu^{1/2}} \left[1+\frac{x^2}{\nu}\right]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$ if $\nu &gt; 1$</td>
<td>$\frac{\nu}{\nu-2}$ if $\nu &gt; 2$</td>
</tr>
<tr>
<td>Uniform</td>
<td>Un($a$, $b$)</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>We($\alpha$, $\beta$)</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\Gamma\left(\frac{\nu+1}{\alpha}\right)$</td>
<td>$\Gamma\left(\frac{\nu+2\alpha+1}{\alpha}\right)$</td>
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