Final Examination

STA 711: Probability & Measure Theory

Sunday, 2020 Nov 22, 2:00 – 5:00 pm

This is a time-limited open-book exam. You may use your notes, class notes, or the text, but do not discuss the problems with anyone (in the class or not) except the TAs and me by e-mail or Zoom. Do not search the internet for help.

You may download and begin the exam at any time of your choosing between **Sun Nov 22 1:30pm EST** (1830 UTC) and **Sun Nov 22 2:30pm EST** (1930 UTC). You then have 3 hours to complete it, and an additional 30 minutes to scan your solutions and upload them as a **single pdf file** using "Tests & Quizzes" in Sakai, no more than 3:30 (= 210 minutes) after your initial download. Thus all exams must be submitted by **Sun Nov 22 6:00pm EST** (2300 UTC).

If a question seems ambiguous or confusing, *please* ask me to clarify it. I will be available by e-mail or zoom on Sun Nov 22 1:30–6:00pm EST. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find. For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, limits, maxima, minima, *etc.*, or unreduced fractions. Wherever possible, **Simplify**. Good luck!

Print Name Clearly: ______________________________

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Problem 1: Let $\Omega = (0, 1]$ with Lebesgue measure on the Borel sets. A few questions about $L_p(\Omega, \mathcal{F}, \mathsf{P})$ and convergence:

a) (5) For which $0 < p \le \infty$ does $X_n := \sqrt{n}\, \mathbf{1}_{\{(0,1/n^3]\}}$ converge in L_p? Prove it.

b) (5) For which $0 \le a < b \le 1$ is the event $E := (a, b] \in \mathcal{F}$ independent of the RV $X := \mathbf{1}_{\{(0,1/2]\}}$? Give *all* solutions.

c) (5) For which $0 \le a < b \le 1$ is the event $E := (a, b] \in \mathcal{F}$ independent of the RV $Y(\omega) := \omega(1 - \omega)$? Give *all* solutions.

d) (5) For which $\alpha > 0$ and $p > 0$ is the RV $Z(\omega) = \omega^{-1/\alpha}$ in L_p? Give *all* solutions.

Problem 2: Let $\Omega = \mathbb{N} := \{1, 2, \cdots\}$ and $\mathcal{F} = 2^\Omega$.

a) (10) Construct a probability measure P on $(\Omega, \mathcal{F})$ and an RV X on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathsf{P}[a < X < b] > 0$ for every open interval $(a, b) \subset \mathbb{R}$.

b) (10) Construct a probability measure Q on $(\Omega, \mathcal{F})$ and an RV Y on $(\Omega, \mathcal{F}, \mathsf{Q})$ such that Y has the $\mathsf{Po}(2)$ probability distribution (Poisson with mean two— see Pg. 12)

Problem 3: Let $\Omega = \mathbb{N} := \{1, 2, \cdots\}$ and $\mathcal{F} = 2^\Omega$. Define probability measures P, Q for events $A \subset \Omega$ by

$$\mathsf{P}(A) := \sum_{\omega \in A} 2^{-\omega} \qquad \mathsf{Q}(A) := \frac{6}{\pi^2} \sum_{\omega \in A} \frac{1}{\omega^2}.$$

a) (5) For which $0 < p \le \infty$ is the RV $X(\omega) = \omega$ in $L_p(\Omega, \mathcal{F}, \mathsf{P})$?

b) (5) For which $0 < p \le \infty$ is the RV $X(\omega) = \omega$ in $L_p(\Omega, \mathcal{F}, \mathsf{Q})$?

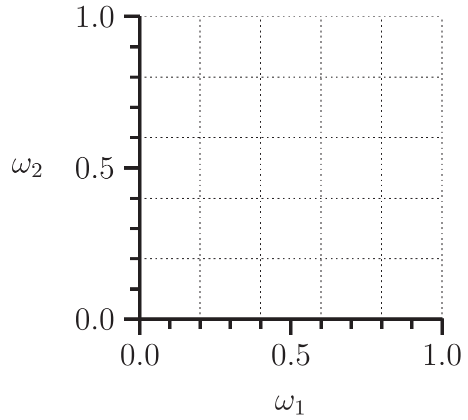
c) (5) Find the Radon-Nikodym derivative
$H(\omega) := \dfrac{\mathsf{P}(d\omega)}{\mathsf{Q}(d\omega)} =$

d) (5) With H from part c) above and $E := \{2, 4, 6, 8, \cdots\}$ the set of all even numbers, evaluate the integral below. Simplify!
$\displaystyle\int_{\mathbb{R}} H(\omega)\mathbf{1}_E(\omega)\mathsf{Q}(d\omega) =$

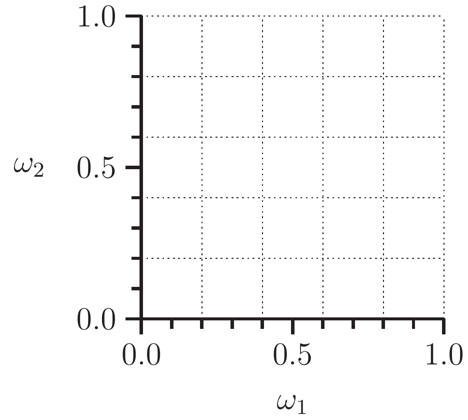
Problem 4: Let $\Omega = (0,1]^2 = \{(\omega_1, \omega_2) : \; 0 < \omega_j \le 1\}$ with Lebesgue measure P on the Borel sets $\mathcal{F}$, and consider the random variables

$$X(\omega) := \omega_1 \qquad Y(\omega) := \omega_2 \qquad S(\omega) := X(\omega) + Y(\omega) \qquad R(\omega) := X(\omega)/Y(\omega)$$

a) (4) Sketch a non-null event $A \in \sigma(R)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(S)$. No explanation is needed.

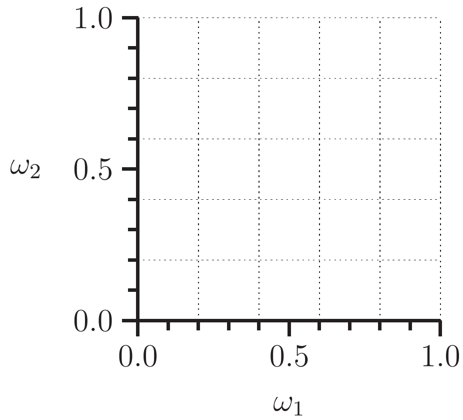


b) (4) Sketch a non-null event $B \in \sigma(S)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(R)$. No explanation is needed.

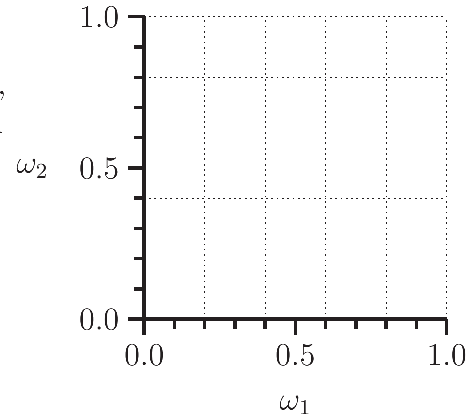


Problem 4 (cont'd): Recall $\Omega = (0,1]^2$ with $\mathsf{P}(d\omega) = d\omega_1\, d\omega_2$, while $X(\omega) := \omega_1$, $Y(\omega) := \omega_2$, $S := X + Y$, and $R := X/Y$. No explanation is needed.

c) (4) Sketch and label **independent** events $D \in \sigma(R)$ and $E \in \sigma(S)$ that are non-trivial– *i.e.*, have probabilities $0 < \mathsf{P}(D), \mathsf{P}(E) < 1$. No explanation is needed.



d) (8) Sketch and label the events $F := \{\omega : \; 0 < \omega_1 \le \omega_2 \le 1\}$ and $G := \{\omega : \; 0 < \omega_1 \le \frac{1}{2}, \; 0 < \omega_2 \le 1\}$, and compute $\mathsf{E}[\mathbf{1}_F \mid \sigma(G)]$. No need to prove anything, just compute the conditional expectation. Show your work.



$\mathsf{E}[\mathbf{1}_F \mid \sigma(G)] =$

Problem 5: Let P be Lebesgue measure on the Borel sets $\mathcal{F}$ of $\Omega = (0, 1]$. Be very explicit for each of the examples below— *e.g.*, give the value of each requested RV at each $\omega \in \Omega$. All RVs must be real-valued.

a) (5) Give an example of an event $A \in \mathcal{F}$ and random variable W on $(\Omega, \mathcal{F}, \mathsf{P})$ that are independent and non-trivial— no constant RVs or null events.

b) (5) Give an example of a random variable X on $(\Omega, \mathcal{F}, \mathsf{P})$ that is almost-surely finite but not essentially bounded, if possible; if not, explain.

c) (5) Give an example Y of a random variable in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ that is not in $L_2(\Omega, \mathcal{F}, \mathsf{P})$, if possible; if not, explain.

d) (5) Give an example Z of a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ whose distribution is neither continuous nor discrete, if possible; if not, explain.

Problem 6: Let $\{X_n\}$ and Y be real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ and for $n, k \in \mathbb{N}$ set $A_{n,k} := \big\{\omega : \ |X_n(\omega) - Y(\omega)| > \frac{1}{k}\big\}$.

a) (5) Give the exact conditions on $A_{n,k}$ for $X_n \to Y$ *a.s.*

b) (5) Give the exact conditions on $A_{n,k}$ for $X_n \to Y$ *pr.*

c) (5) Use your expressions above to prove that almost-sure convergence implies convergence in probability.

d) (5) Prove that $\cos(X_n) \to \cos(Y)$ in $L_2(\Omega, \mathcal{F}, \mathsf{P})$ if $X_n \to Y$ *pr.*

Problem 7: Let $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be iid exponentially-distributed with mean 1, and set $X_n^* := \max\{X_1, ..., X_n\}$ and $S_n := \sum_{1 \le j \le n} X_j$.

a) (6) Show that $(X_n^* - \log n)$ converges in distribution as $n \to \infty$ and find the limiting DF $G(z) := \lim_{n \to \infty} \mathsf{P}\big[(X_n^* - \log n) \le z\big]$ for every $z \in \mathbb{R}$.

b) (4) Find the approximate median M_n for X_n^*, so $\mathsf{P}[X_n^* \le M_n] \approx 1/2$.

c) (6) By SLLN, $S_n/n \to 1$ *a.s.* Prove that also $S_n/n \to 1$ in L_2.

d) (4) Find the smallest bounds $a_n > 0$ and $b_n > 0$ that ensure

$$X_n^* \le a_n S_n \qquad S_n \le b_n X_n^*$$

$a_n =$ ____________ $\qquad b_n =$ ____________

Problem 8: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F For any real number $x \neq 1$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

b) T F If an RV $X \in L_2$ and event A are independent then $\mathsf{E}\big[X\mathbf{1}_A\big] = \mathsf{E}[X] \cdot \mathsf{P}[A]$ and $\mathsf{E}\big[\,|X\mathbf{1}_A|\,\big] = \mathsf{E}[\,|X|\,] \cdot \mathsf{P}[A]$.

c) T F If $\sup_{\omega \in \Omega} |X_n - X| \to 0$ then $X_n \to X$ in L_1.

d) T F If $\mathsf{P}[X = x] = 0$ for every $x \in \mathbb{R}$ then X has a continuous distribution.

e) T F If $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $Y = \mathsf{E}[X \mid \mathcal{G}]$ for some sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, then $|Y| \geq \mathsf{E}\big[\,|X| \mid \mathcal{G}\big]$ *a.s.*

f) T F If $X_n \Rightarrow X$ (convergence in distribution) then, along some subsequence n_k, $X_{n_k} \to X$ almost-surely.

g) T F If $X_n \Rightarrow X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g(X_n) \Rightarrow g(X)$.

h) T F If $\sigma(X) \subset \sigma(Y)$ for $X, Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ then $\mathsf{E}[X \mid Y]$ is the constant RV with value $\mathsf{E}[X]$.

i) T F If $\mathsf{E}[e^{i\omega X}] = \cos(\omega)$ then $\mathsf{P}[X = 1] = 1/2$.

j) T F If $X = \phi(Y)$ for some Borel function ϕ then $\sigma(X) = \sigma(Y)$.

Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	