Midterm Examination I

STA 711: Probability & Measure Theory

75 Minutes starting Thursday, 2020 Sep 24, 1:45pm

This is a time-limited open-book exam. You may use your notes, class notes, or the text, but do not discuss the problems with anyone (in the class or not) except the TAs and me (by e-mail, Zoom, or a "private" Piazza message), and do not use the internet for help.

You may download and begin the exam at any time of your choosing between **Thu Sep 24 1:45pm** and **Fri Sep 25 1:45pm**. You then have 75 minutes to complete it, and an additional 30 minutes to scan your solutions and upload them as a **single pdf file** using Sakai, no more than 1:45 (= 75+30 minutes) after your initial download.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, limits, maxima, minima, *etc.*, or unreduced fractions. Wherever possible, **Simplify**. Good luck!

Print Name Clearly: ________________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: The probability space $\Omega = \{a, b, c, d\}$ has just four points, with σ-algebra $\mathcal{F} := \sigma\big\{\{a\},\ \{b\}\big\}$ and probability measure P assigning equal probabilities

$$\mathsf{P}(\{a\}) = p/2 = \mathsf{P}(\{b\})$$

to the indicated singletons, for some number $0 \le p \le 1$.

a) (5) How many elements does $\mathcal{F}$ have?

b) (5) Is the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ *complete*? If this depends on the value of $p \in [0, 1]$, explain.

c) (5) If a second probability measure Q on $(\Omega, \mathcal{F})$ also satisfies $\mathsf{Q}(\{a\}) = p/2 = \mathsf{Q}(\{b\})$, must P and Q agree on all of $\mathcal{F}$? Why, or why not?

d) (5) Describe all random variables X on $(\Omega, \mathcal{F}, \mathsf{P})$.

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with Lebesgue measure $\mathsf{P}(d\omega) = d\omega$ and set $Y_1(\omega) := 1/\omega$ and $Y_2(\omega) := 1/(1-\omega)$ for $\omega \in \Omega$. Let $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ converge at each ω to a limiting function $X : \Omega \to \mathbb{R}$ (which may or may not be in L_1), *i.e.*, let $X_n(\omega) \to X(\omega)$.

a) (5) If $|X_n| \le Y_1$, does it follow that $\mathsf{E}[X_n] \to \mathsf{E}[X]$? If so, prove it; if not, give an example where convergence fails.

b) (5) If $|X_n| \le (Y_1 \wedge Y_2)$ (the pointwise minimum), does it follow that $\mathsf{E}[X_n] \to \mathsf{E}[X]$? If so, prove it; if not, give an example where convergence fails.

Problem 2 (cont'd): Still $Y_1(\omega) := 1/\omega$, $Y_2(\omega) := 1/(1-\omega)$, $\mathsf{P}(d\omega) = d\omega$, and $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega := (0,1]$.

c) (5) If $|X_n| \le Y_1 Y_2$ and $0 \le X_n \le X_{n+1}$ for each n, does it follow that $\mathsf{E}[X_n] \to \mathsf{E}[X]$? If so, prove it; if not, give an example where convergence fails.

d) (5) Let $\mathsf{Q}(d\omega) = 6\omega(1-\omega)\,d\omega$ be the $\mathsf{Be}(2,2)$ probability measure on $(\Omega, \mathcal{F})$. If $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{Q})$ converge pointwise to a limit X, and if $|X_n| \le (Y_1 + Y_2)$, does it follow that $X_n \to X$ in $L_1(\Omega, \mathcal{F}, \mathsf{Q})$? If so, prove it; if not, give an example where convergence fails.

Problem 3: Let $\{X_n\} \overset{\text{ind}}{\sim} \mathsf{Ex}(n)$ be independent exponential-distributed[1] random variables with $\mathsf{P}[X_n > x] = \exp(-n\,x)$ for $x > 0$ and $n \in \mathbb{N}$. For $n \in \mathbb{N}$ set:

$$S_n := \sum_{m=1}^{n} X_m \qquad\qquad T_n := \sum_{m=1}^{n} \mathbf{1}_{\{X_m>2\}}.$$

Show your work in finding:

a) (4) For every $0 < p \le \infty$ and $n \in \mathbb{N}$, find[2]:
$\mathsf{E}|X_n|^p =$

b) (4) Does the Dominated Convergence Theorem apply to S_n? If so, what does it say? If not, why not?

c) (4) Does the Dominated Convergence Theorem apply to T_n? If so, what does it say? If not, why not?

[1] Recall there is a page with dist'ns pdfs/pmfs, means, variances, *etc.* on *p.* 10.
[2] Recall $\Gamma(z) := \int_0^\infty x^{z-1}\,e^{-x}\,dx < \infty$ for $z > 0$; for $z \in \mathbb{N}$, $\Gamma(z) = (z-1)!$.

Problem 3 (cont'd): Still $\{X_n\} \overset{\text{ind}}{\sim} \mathsf{Ex}(n)$ w/ $\mathsf{P}[X_n > x] = e^{-n\,x}$ for $x > 0$, $S_n := \sum_{m=1}^n X_m$, and $T_n := \sum_{m=1}^n \mathbf{1}_{\{X_m > 2\}}$.

d) (4) What is the probability that T_n converges to a finite limit? Prove your answer.

e) (4) Find the pdf $g(y)$ for the random variable $Y := 1/X_2$ correctly for all $y \in \mathbb{R}$— or, if no such pdf exists, explain why.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}, \lambda\big)$ be the unit interval with its Borel sets and Lebesgue measure. Let $\{A_n\}$ be independent events in $\mathcal{F}$ with probabilities $\mathsf{P}(A_n) = 1/n$, and set $B_n := (0, 1/n] \in \mathcal{F}$.

a) (5) What is the prob. that **all but finitely-many** $\{A_n\}$ occur? Why?
$\mathsf{P}[\liminf A_n] =$

b) (5) What is the probability that **infinitely-many** $\{A_n\}$ occur? Why?
$\mathsf{P}[\limsup A_n] =$

c) (5) What is the probability that infinitely-many $\{B_n\}$ occur? Why?
$\mathsf{P}[\limsup B_n] =$

d) (5) What is the probability that infinitely-many $\{A_{n^2}\}$ occur? Why?
$\mathsf{P}[\limsup A_{n^2}] =$

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think a question seems ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If $\mathcal{G}_X := \sigma(X)$ and $\mathcal{G}_Y := \sigma(Y)$ for RVs X, Y then $X + Y$ is measurable over the σ-algebra $\mathcal{G}_X \cap \mathcal{G}_Y$.

b) T F If $\{X_n\}$ are discrete RVs on $(\Omega, \mathcal{F}, \mathsf{P})$ then so is $\sup_n X_n$.

c) T F If $\{X_n\}$ satisfy $\|X_n\|_1 \le 10$ and $X_n(\omega) \to X(\omega)$ for each $\omega \in \Omega$ then $\|X\|_1 \le 10$.

d) T F If $\{A, B, C\}$ are independent events then the RVs $\mathbf{1}_A$ and $\mathbf{1}_{B\cup C}$ are independent.

e) T F For any RV X and any $\epsilon > 0$ there exist a discrete RV Y and an absolutely continuous RV Z with $\mathsf{P}\big[|X - Y| < \epsilon\big] = 1 = \mathsf{P}\big[|X - Z| < \epsilon\big]$.

f) T F There are only countably many finite subsets of $\mathbb{Z} := \{..., -1, 0, 1, 2, ...\}$.

g) T F For every real-valued RV X, the function $f(\omega) := \mathsf{E}[\cos(\omega X)]$ is a well-defined, finite, and continuous function of $\omega \in \mathbb{R}$.

h) T F Every field on a finite set Ω is also a σ-field.

i) T F For *finite* RVs $\{X_n\}$, always $\liminf_{n\to\infty} X_n < \infty$.

j) T F For every RV X and $p \ge 1$, $\mathsf{P}[X > 10] \le \|X\|_p / 10$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	