Midterm Examination II

STA 711: Probability & Measure Theory

75 Minutes starting Thursday, 2020 Oct 29, 1:45pm

This is a time-limited open-book exam. You may use your notes, class notes, or the text, but do not discuss the problems with anyone (in the class or not) except the TAs and me (by e-mail, Zoom, or a "private" Piazza message), and do not use the internet for help.

You may download and begin the exam at any time of your choosing between **Thu Oct 29 1:45pm** and **Fri Oct 30 1:45pm**. You then have 75 minutes to complete it, and an additional 30 minutes to scan your solutions and upload them as a **single pdf file** using Sakai, no more than 1:45 (= 75+30 minutes) after your initial download.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, limits, maxima, minima, *etc.*, or unreduced fractions. Wherever possible, **Simplify**. Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be iid standard uniform RVs on some space $(\Omega, \mathcal{F}, \mathsf{P})$. In each part below, indicate in which (if any) sense(s) the sequence $\{Y_n\}$ converges to **zero**. No explanations are necessary.

a) (4) $Y_n := e^{-nX_n}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ in dist.

b) (4) $Y_n := e^{-n-X_n}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ in dist.

c) (4) $Y_n := \prod_{1\le j\le n} X_j$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ in dist.

d) (4) $Y_n := \sum_{1\le j\le n} \frac{1-2X_j}{n}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ in dist.

e) (4) $Y_n := n \min_{1\le j\le n} X_j$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ in dist.

Problem 2: Let $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(\theta)$ be independent Exponentially-distributed[1] random variables, and let $S_n := \sum_{j=1}^n X_j$ denote their partial sum.

a) (6) A common data-based statistical estimator for θ is the random variable $\hat{\theta}_n := n/S_n$. Prove that this converges almost-surely to θ as $n \to \infty$. This property of an estimator is called "strong consistency".

b) (6) The Bayesian posterior mean of θ for a "conjugate" $\mathsf{Ga}(\alpha, \beta)$ prior distribution is $\bar{\theta}_n = (\alpha + n)/(\beta + S_n)$. Prove that this too converges almost-surely to θ as $n \to \infty$, for any $\alpha, \beta > 0$.

[1] Recall that Pg. 10 shows pdfs/pmfs, means, variances, *etc.* for most commonly-encountered distributions.

Problem 2 (cont'd): Still $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(\theta)$ and $S_n := \sum_{j=1}^n X_j$.

c) (2) Verify that the Gamma distribution $Y \sim \mathsf{Ga}(\alpha, \beta)$ has moments $\mathsf{E}[Y^p] = \beta^{-p}\Gamma(\alpha+p)/\Gamma(\alpha)$ for all $p > -\alpha$, and $\mathsf{E}[Y^p] = \infty$ for $p \le -\alpha$, for all $\alpha > 0$ but just for the special case of $\beta = 1$ (even though it's true for all $\beta > 0$), *i.e.*, for $Y \sim \mathsf{Ga}(\alpha, 1)$ and $p \in \mathbb{R}$ find $\mathsf{E}[Y^p]$.

d) (6) It turns out that S_n has a Gamma distribution $S_n \sim \mathsf{Ga}(n, \theta)$. Compute the average error $b_n(\theta) := \mathsf{E}[\hat{\theta}_n - \theta]$ for $\hat{\theta}_n := n/S_n$ for each $n \in \mathbb{N}$ and $\theta > 0$. Is $\hat{\theta}_n$ "unbiased", *i.e.*, is $b_n(\theta)$ identically zero? ◯ Yes ◯ No

Problem 3: For $1 \le i \le j \in \mathbb{N}$ set $n := j(j-1)/2 + i$ and

$$X_n := \mathbf{1}_{\left(\frac{i-1}{j}, \frac{i}{j}\right]} \qquad\qquad Y_n := nX_n.$$

(suggestion: write out X_1, X_2, X_3, X_4 explicitly).

a) (4) Does $Y_n \to 0$ $(pr.)$? ◯ Yes ◯ No Why?

b) (4) Does $Y_n \to 0$ $a.s.$? ◯ Yes ◯ No Why?

c) (4) For which $0 < p < \infty$ does $X_n \to 0$ in L_p? Why?

d) (4) For which $0 < p < \infty$ does $Y_n \to 0$ in L_p? Why?

e) (4) Does $X_n \to 0$ in L_∞? Prove it. ◯ Yes ◯ No Why?

Problem 4: Let $\{X_n\}, \{Y_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be independent exponentially-distributed RVs, and set

$$Z_n := (X_n - Y_n) \qquad S_n := \sum_{m=1}^{n} Z_m.$$

a) (4) Find the characteristic functions (ch.f.s) for X_n and Z_n. *Simplify!*
$\phi_X(\omega) := \mathsf{E}\big[e^{i\omega X_n}\big] =$ $\qquad\qquad$ $\phi_Z(\omega) := \mathsf{E}\big[e^{i\omega Z_n}\big] =$

b) (4) Find the characteristic function for S_n. *Simplify!*
$\mathsf{E}\big[e^{i\omega S_n}\big] =$

c) (4) Find the indicated means and variances [2]:

$\mathsf{E}[X_n] =$ $\qquad$ $\mathsf{E}[Z_n] =$ $\qquad$ $\mathsf{E}[S_n] =$
$\mathsf{V}[X_n] =$ $\qquad$ $\mathsf{V}[Z_n] =$ $\qquad$ $\mathsf{V}[S_n] =$

[2] Remember Pg. 10

Problem 4 (cont'd): Still $\{X_n\}, \{Y_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$, $Z_n := (X_n - Y_n)$, and $S_n := \sum_{m=1}^{n} Z_m$.

d) (4) Find z below and justify your answer[3]:
$\mathsf{P}[S_{50} > 10] \approx \Phi(z)$ for $z =$

e) (4) If $N \sim \mathsf{Po}(\lambda)$ has a Poisson distribution, what is the ch.f. for S_N, the sum of a random number of Z_ns?
$\mathsf{E}\big[\exp\big(i\omega S_N\big)\big] =$

[3] $\Phi(z) := \int_0^z \exp(-x^2/2)\,dx/\sqrt{2\pi}$, CDF for the $\mathsf{No}(0,1)$ distribution.

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think a question seems ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If $X_n \to X$ in L_2 then along some subsequence $X_{n_j} \to X$ *a.s.*

b) T F If $|X_n| \le Y_n$ and $Y_n \to 0$ (*pr.*), then $X_n \to 0$ (*pr.*).

c) T F If $\mathsf{P}[A_n] \ge 1 - 3^{-n}$ then $\mathsf{P}[\cap_{n=1}^N A_n] > 1/2$ for all $N \in \mathbb{N}$.

d) T F If $\mathsf{P}[A \cap B \cap C] = \mathsf{P}[A] \cdot \mathsf{P}[B] \cdot \mathsf{P}[C]$ then A, B, C are independent.

e) T F If $\{X_n\}$ are iid but not L_2 then the CLT does not apply.

f) T F If $\{X_n\}$ are independent, then the RV $\limsup_{n\to\infty} X_n$ is constant.

g) T F If $\mathsf{E}|X_n|^4 \le B < \infty$ and $X_n \to X \in L_2$ (*pr.*) then $X_n \to X$ in L_2.

h) T F If $\mathsf{E}[X\mathbf{1}_A] = \mathsf{E}[X] \cdot \mathsf{P}[A]$ then X, A are independent.

i) T F If $\mathsf{E}\big[\exp(|X_\alpha|)\big] \le B < \infty$ then $\{X_\alpha\}$ are UI.

j) T F If X has ch.f. $\phi_X(\omega)$ then $Y := -X$ has ch.f. $\phi_X(-\omega)$.

Blank Worksheet

Another Blank Worksheet

Name	**Notation**	**pdf/pmf**	**Range**	**Mean** μ	**Variance** σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\, q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	
		$f(y) = \alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y \in (\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	$(y = x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 > 2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2 > 4$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0 if $\nu > 1$	$\frac{\nu}{\nu-2}$ if $\nu > 2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x) = \frac{1}{b-a}$	$x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	