Fields and $\sigma$-fields

1. Enumerate the class $\mathcal{F}$ of all $\sigma$-fields $\mathcal{F}$ on the three-point set $\Omega = \{a, b, c\}$ that contain the singleton $\{a\}$, i.e., that satisfy $\mathcal{C} \subset \mathcal{F}$ for $\mathcal{C} := \{\{a\}\}$. What is $\sigma(\mathcal{C})$?

2. Prove that for any two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on any set $\Omega$, the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a field.

3. Find a set $\Omega$ and two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega$ for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a field.

4. Suppose a collection $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of $\sigma$-fields on a set $\Omega$ satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\bigcup \mathcal{F}_j$ is a field? (the answer is “yes” — show why)

5. Under the same conditions, must $\bigcup \mathcal{F}_j$ be a $\sigma$-field? (this one is “no” — find a counterexample. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\cup_n A_n \notin \mathcal{F}_j$ for every $j$, hence $\cup_n A_n \notin \cup_j \mathcal{F}_j$).
Dyadic Rational Probability Spaces

For problems 6–9, let $\Omega = \mathbb{Q}_2 := \{j/2^n : j \in \{1, 2, \cdots, 2^n\}, n \in \mathbb{N}\}$ be the dyadic rational numbers in the half-open unit interval, and let

$$\mathcal{C} = \{(0, b] \cap \mathbb{Q}_2 : b \in \mathbb{Q}_2, 0 < b \leq 1\}$$

(1)
denote the collection of half-open intervals of dyadic rationals $(0, b] = \{q \in \mathbb{Q}_2 : 0 < q \leq b\}$ with left endpoint zero. Every $\Omega$ on this page contains only dyadic rational numbers.

Recall that a real-valued set function $P$ on a $\sigma$-algebra $\mathcal{G}$ of subsets of a space $\Omega$ is a “probability measure” (PM) if and only if it satisfies the three rules:

- $(\forall A \in \mathcal{G}) \ P(A) \geq 0$;
- $(\forall \{A_i\} \subset \mathcal{G}, \ A_i \cap A_j = \emptyset) \ P(\bigcup A_i) = \sum P(A_i)$;
- $P(\Omega) = 1$.

6. Let $n \in \mathbb{N}$ be a FIXED positive integer (like three) and set

$$\mathcal{B}_n := \{(0, j/2^n], j \in \{0, 1, \cdots, 2^n\}\},$$

the collection of half-open intervals in $\Omega$ of dyadic rationals from zero up to an integral multiple of $2^{-n}$. Describe the elements of the $\sigma$-field

$$\mathcal{F}_n := \sigma(\mathcal{B}_n)$$
generated by $\mathcal{B}_n$, for fixed $n \in \mathbb{N}$. How many elements does $\mathcal{B}_n$ have? How many distinct elements does $\mathcal{F}_n$ have? What are they? Suggestion: Try $\mathcal{B}_0, \mathcal{B}_1$ and $\mathcal{B}_2$ first, by hand. Is there a partition that generates $\mathcal{F}_n$?

7. What is the field $\mathcal{F}^* := \mathcal{F}(\mathcal{C})$ of subsets of $\mathbb{Q}_2$ generated by the class $\mathcal{C}$ of Eqn (1)? (hint: Do problems (4) and (6) first). Try to describe it in just a few words, without using any symbols besides $\mathbb{Q}_2$. Don’t just echo the definition!

8. Describe simply and clearly in no more than five words or symbols (seriously, three should be enough) the $\sigma$-field $\mathcal{F} := \sigma(\mathcal{C})$ of subsets of $\mathbb{Q}_2$ generated by $\mathcal{C}$. Don’t just echo the definition!

9. Define a set function $\lambda_0$ on $\mathcal{C}$ by

$$\lambda_0(\ (0, b] \ ) = b$$

Show that there does not exist a probability measure $\lambda$ on $(\mathbb{Q}_2, \mathcal{F})$ that extends $\lambda_0$, i.e., one for which $\lambda(\ (0, b] \ ) = b$ for all $b \in \mathbb{Q}_2$ (Hint: Exactly what does the function $F(x) := \lambda(\ (0, x] \ ), 0 \leq x \leq 1$ look like near $x \in \mathbb{Q}_2$, for any PM $\lambda$ on $\mathbb{Q}_2$?).