Random variables

1. Let $(\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)$ for Lebesgue measure $\lambda$ on the Borel sets of the unit interval. For $\omega \in \Omega$ define:

$$
X_1(\omega) := \min(\omega, 0.6) \quad X_2(\omega) := 1_{(0, 1/3]}(\omega) \quad X_3(\omega) := \sqrt{\omega}
$$

Plot each of the CDFs $F_k(x) := P[X_k \leq x], x \in \mathbb{R}$, and describe explicitly the $\sigma$-algebras $\mathcal{F}_k := \sigma(X_k)$.

2. Let $X$ be a random variable with CDF $F(x) := P(X \leq x)$. Set $Y := F(X)$. If $X$ has a continuous distribution (i.e., if $F$ is a continuous function), show that $Y$ is a random variable and that $Y$ has a uniform distribution on $[0, 1]$. Warning: $F(x)$ may not be strictly increasing, and so may not be one-to-one; also it may not be differentiable.

3. A random variable $Y$ is real-valued if $Y(\omega) \in \mathbb{R}$ for every $\omega \in \Omega$, and is bounded if there is a fixed finite number $0 \leq B < \infty$ for which $|Y(\omega)| \leq B$ for all $\omega \in \Omega$. Give an example of a real-valued random variable $X$ that is not bounded.

4. Let $X$ be a real valued random variable (so $P[|X| < \infty] = 1$) with CDF $F(x)$. For each $\epsilon > 0$, construct a bounded random variable $Y_\epsilon$ such that

$$
P(X \neq Y_\epsilon) < \epsilon.
$$

Measurable functions

5. Set $\mathcal{S} := \sigma\{(-\infty, 0]\}$ on $\Omega := \mathbb{R}$. Describe all Borel functions $f : \Omega \to \mathbb{R}$ that are $\mathcal{S}\backslash\mathcal{B}$-measurable.

6. If $X$ is a real-valued random variable on any probability space $(\Omega, \mathcal{F}, P)$, then show that $|X|$ is also a random variable. Show by an example that the converse need not be true (Hint: A finite $\Omega$ will suffice).

7. Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces, and let $X : \Omega \to E$ be any function. Show that $\mathcal{H} := \{B \in \mathcal{E} : X^{-1}(B) \in \mathcal{F}\}$ is a $\sigma$-algebra.

8. Again let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces and $X : \Omega \to E$ any function. If $\mathcal{E} = \sigma(\mathcal{C})$ is generated by a class $\mathcal{C} \subset \mathcal{E}$ of sets, show that $X$ is $\mathcal{F}\backslash\mathcal{E}$-measurable if and only $X^{-1}(C) \in \mathcal{F}$ for each $C \in \mathcal{C}$.

9. Let $\mathcal{F}_X := \sigma(X)$ be the $\sigma$-algebra generated by the function $X(\omega) := \omega^2$ on $\Omega = \mathbb{R}$. Is the set $A = (-\infty, 0]$ in $\mathcal{F}_X$? How about $B = [-4, 4]$? Why?
10. Let \( \{X_n, n \geq 0\} \) be real-valued random variables on \((\Omega, \mathcal{F}, P)\) that satisfy
\[
\limsup_{n \to \infty} X_n(\omega) = +\infty
\]
for every \( \omega \in \Omega \), and let \( B < \infty \) be a real number. Prove that the integer-valued quantity
\[
\tau(\omega) := \inf\{n \geq 0 : X_n(\omega) \geq B\}
\]
is a random variable.

**Extra credit:** Prove that \( X_\tau \) is also a random variable.

**Random Variables and \( \sigma \)-Algebras**

11. Let \( \Omega = \mathbb{Z}_+ := \{0, 1, 2, \ldots\} \) be the non-negative integers, and set
\[
X(\omega) := \omega \quad Y(\omega) := \omega 1_E(\omega) \quad X_n(\omega) := \min\{n, \omega\}
\]
for each natural number \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \), where \( E := \{0, 2, 4, 6, \ldots\} \) denotes the even nonnegative integers and \( 1_E(\cdot) \) its indicator function.

(a) Find \( \mathcal{F} := \sigma(X) \).

(b) Set \( \mathcal{F}_n := \sigma(X_n) \). Find \( \mathcal{F}_2 \) explicitly, by enumerating its elements.

(c) Describe \( \mathcal{G} := \sigma(Y) \). Give a set \( A \in \mathcal{F} \) that is not in \( \mathcal{G} \), and a set \( B \) (other than \( \emptyset \) and \( \Omega \)) that is in \( \mathcal{G} \).

(d) Is \( \mathcal{H} := \bigcup \mathcal{F}_n \) a \( \sigma \)-algebra? Prove your answer.

(e) Is \( \mathcal{H} := \bigcup \mathcal{F}_n \) the same as \( \mathcal{F} \)? If so, prove it; if not, find some event \( A \in \mathcal{F} \setminus \mathcal{H} \).