Sta 711: Homework 4

Expectation

Feel free to use the result of one problem in your solution to a subsequent problem.

1. Let \( X := (X_1, X_2) \) be distributed uniformly over the triangle in \( \mathbb{R}^2 \) with vertices \((-1, 0), (1, 0), (0, 1)\). Evaluate \( \mathbb{E}(X_1 + X_2) \) (no need to approximate by simple functions, just find the value and show how you did it).

2. Let \( X \geq 0 \) be a nonnegative random variable on \((\Omega, \mathcal{F}, \mathbb{P})\) and, for \( n \in \mathbb{N} \), set
   \[ X_n(\omega) := \min \left( 2^n, 2^{-n} \left\lfloor 2^n X(\omega) \right\rfloor \right). \]
   Prove that \( X_n \) is simple (how many distinct values can it take on?) and \( X_n \nearrow X \). Note you must show both monotonicity and convergence. For \( \omega \in \Omega \) and \( \epsilon > 0 \), how big must \( n \) be to ensure \( |X - X_n| < \epsilon \)?

3. Suppose \( X \in L_1(\Omega, \mathcal{F}, \mathbb{P}) \), i.e., \( \mathbb{E}|X| < \infty \). Show that\footnote{The “expectation of a random variable \( X \) over an event \( A \)” can be written in many ways, including \( \int_A X d\mathbb{P} = \mathbb{E}[X1_A] = \int_A X(\omega)\mathbb{P}(d\omega) \).}
   \[ \int_{|X| > m} X d\mathbb{P} \to 0 \quad \text{as} \quad m \to \infty. \]

4. Let \( \{A_n\} \) denote a sequence of events such that \( \mathbb{P}(A_n) \to 0 \) as \( n \to \infty \) and let \( X \in L_1 \). Show that
   \[ \mathbb{E}[X1_{A_n}] = \int_{A_n} X d\mathbb{P} \to 0 \]
   Hint: Use problem 3. Warning: these conditions do not imply that \( X(\omega)1_{A_n}(\omega) \to 0 \).

5. Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and define a distance measure \( d \) on \( \mathcal{F} \) by \( d(A, B) := \mathbb{P}(A \triangle B) \) where (as usual) \( A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \) denotes the symmetric difference. Show that, if \( \{A_n\} \subset \mathcal{F} \) and \( A \in \mathcal{F} \) satisfy \( d(A_n, A) \to 0 \), then
   \[ \int_{A_n} X d\mathbb{P} \to \int_A X d\mathbb{P} \]
   for every \( X \in L_1(\Omega, \mathcal{F}, \mathbb{P}) \). Hint: Use problem 4.
Convergence Theorems

6. Let $X \geq 0$ be a non-negative random variable. Define sequences of random variables $X_n$ and of extended real numbers $0 \leq S_n \leq \infty$ for positive integers $n \in \mathbb{N}$ by:

$$X_n := \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{\{k < 2^n X \leq k+1\}}$$

$$S_n := \mathbb{E}X_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{P}\left\{ \frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right\}$$

Is $X_n$ “simple”? What is $\lim_{n \to \infty} S_n$? Justify your answers.

7. Define a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}, \lambda)$ by

$$X_n := \frac{n}{\log(n+1)} 1_{(0,\frac{1}{n}]} \quad n \in \mathbb{N}.$$ 

Show that $\mathbb{P}[X_n \to 0] = 1$, and that $\mathbb{E}(X_n) \to 0$. Also show that the Dominated Convergence Theorem does not apply to this example. Why?

8. Let $\{Y_n\}$ be a sequence of random variables for $n \in \mathbb{N}$ with

$$\mathbb{P}(Y_n = -n^3) = \mathbb{P}(Y_n = +n^3) = \frac{1}{2n^2}, \quad \mathbb{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

One can (but you don’t have to) use the Borel-Cantelli lemma to show that $Y_n \to 0$ a.s. Compute $\lim_{n \to \infty} \mathbb{E}(Y_n)$ and $\lim_{n \to \infty} \mathbb{E}(|Y_n|)$. Is the Dominated Convergence Theorem applicable? Why or why not?

9. Let $\{X_n\}, X$ be random variables with $0 \leq X_n \to X$. If $\sup_n \mathbb{E}(X_n) \leq K < \infty$, show that $X \in L_1$ and $\mathbb{E}(X) \leq K$. Does $X_n \to X$ in $L_1$?

**Domination**

10. Let $\{X_n\}$ be a sequence of random variables. We have seen that $\{X_n\}$ is dominated by some integrable $Y$, i.e., $|X_n| \leq Y$ and $\mathbb{E}Y < \infty$, if and only if

$$\mathbb{E}\left(\sup_{n \in \mathbb{N}} |X_n|\right) < \infty \quad (1)$$

Thus, (1) is exactly equivalent to domination in Lebesgue’s sense (but Lebesgue’s domination is often easier to verify). Does the condition

$$\sup_{n \in \mathbb{N}} \mathbb{E}(|X_n|) < \infty \quad (2)$$

imply (1)? Or is it implied by (1)? For each direction (1$\Rightarrow$2 and 2$\Rightarrow$1), give either a proof or a counter-example.