Convergence

1. Let \( X \) be a strictly positive random variable. Show that:
   
   (a) \( \lim_{n \to \infty} n \mathbb{E}(\frac{1}{X}1_{[X>n]}) = 0. \)
   
   (b) \( \lim_{n \to \infty} n^{-1} \mathbb{E}(\frac{1}{X}1_{[X>n-1]}) = 0. \)

2. Let \( X \sim \text{Un}(0, 4] \) be uniformly distributed on the interval \((0, 4]\), and set \( Y := \frac{1}{X} \) and \( Z := \log(4Y) \). Suggestion: First find out what is the distribution of \( Z \), by computing \( P[Z > z] \) for \( z \in \mathbb{R} \). Use \( \varphi(x) := |x| \) for the Markov inequality questions.
   
   (a) What bound does Markov’s inequality give for \( P[X > 3] \)?
   
   (b) What bound does Chebychev’s inequality give for \( P[|X - 2| > 1] \)?
   
   (c) What bound does Markov’s inequality give for \( P[Y > 1] \)?
   
   (d) What bound does Markov’s inequality give for \( P[Z > 2] \)?
   
   (e) What are the exact values of \( P[X > 3] \), \( P[|X - 2| > 1] \), \( P[Y > 1] \), and \( P[Z > 2] \)?

3. Let \( A \) and \( B \) be events in \((\Omega, \mathcal{F}, P)\) with probabilities \( a = P(A) \) and \( b = P(B) \) respectively. Show that \( P(A \cap B) \leq \sqrt{ab} \).

4. Suppose \( \{X_n\}, X \) are real valued RVs defined on a probability space \((\Omega, \mathcal{F}, P)\) and that \( X_n(\omega) \to X(\omega) \) for all \( \omega \in \Omega \). Show that for every \( \epsilon > 0 \), there is an event \( \Lambda_\epsilon \) with \( P(\Lambda_\epsilon) < \epsilon \) and

   \[
   \sup_{\omega \in \Lambda_\epsilon} |X(\omega) - X_n(\omega)| \to 0 \quad \text{as} \quad n \to \infty.
   \]

   Thus the convergence is uniform except on an arbitrarily small set. (For more on this result, called Egorov’s Theorem, see page 89 of the text.)

5. For a random variable \( X \), \( 1 < p < q < \infty \), show\(^1\) that

   \[
   0 \leq \|X\|_1 \leq \|X\|_p \leq \|X\|_q \leq \|X\|_\infty
   \]

6. For \( 1 < p < q < \infty \), show that

   \[
   L_\infty \subset L_q \subset L_p \subset L_1
   \]

   where \( L_p := \{X : \|X\|_p < \infty\} \).

\(^1\)Hint: Jensen’s inequality may help for some parts
7. The “Moment Generating Function” (MGF) of a real-valued random variable \(X\) (or of its distribution \(\mu(dx)\)) is the extended real-valued function \(M_X(t) := \mathbb{E}\exp(tX) = \int_{\mathbb{R}} e^{tx} \mu(dx)\) of \(t \in \mathbb{R}\). Show that a nonnegative random variable \(X \geq 0\) is in \(L_1\) if there exists some \(t > 0\) for which \(M_X(t) < \infty\). Show that the converse may fail—i.e., there exist \(X \geq 0\) in \(L_1\) for which \(M_X(t) = \infty\) for all \(t > 0\).

8. Show that Minkowski’s Inequality fails for \(0 < p < 1\)—i.e., find \((\Omega, \mathcal{F}, \mathbb{P})\) and \(X, Y \in L_p(\Omega, \mathcal{F}, \mathbb{P})\) for which \(\|X + Y\|_p > \|X\|_p + \|Y\|_p\) for some \(0 < p < 1\).