Sta 711: Homework 7

Almost-sure Convergence

1. Let \( \{X_n\} \) be a monotonically increasing sequence of RVs such that \( X_n \to X \) in probability (pr.). Show that \( X_n \to X \) almost surely (a.s.)

2. Let \( \{X_n\} \) be any sequence of RVs. Show that \( X_n \to X \) a.s. if and only if
\[
\sup_{k \geq n} |X_k - X| \to 0 \quad \text{pr.}
\]

3. Let \( \{X_n\} \) be an arbitrary sequence of RVs and set \( S_n := \sum_{i=1}^{n} X_i \). Show that \( X_n \to 0 \) a.s. implies that \( S_n/n \to 0 \) a.s.

In-probability Convergence

4. Let \( \{X_n\} \subset L_2 \) be independent and identically distributed. For each \( \delta > 0 \) show that
\[
nP(|X_1| > \delta \sqrt{n}) \to 0.
\]
Use this to show that the maximum \( \bigvee_{i=1}^{n} |X_i|/\sqrt{n} \to 0 \) pr. Thus, the maximum of \( n \) iid \( L_2 \) random variables grows slower than \( \sqrt{n} \).

5. For random variables \( X, Y \) define
\[
\rho(X, Y) := \inf \{\epsilon > 0 : P(|X - Y| \geq \epsilon) \leq \epsilon\}.
\]
The function \( \rho \) is a metric (you do not have to prove that), i.e., it’s non-negative, symmetric, satisfies the triangle inequality, and vanishes if and only if \( X = Y \) a.s. Show that \( X_n \to X \) pr. if and only if \( \rho(X_n, X) \to 0 \). Thus, convergence in probability is metrizable.\(^1\)

6. Let \( P \) and \( Q \) be two probability measures on \( (\Omega, \mathcal{F}) \) that are “equivalent” in the sense that \( P[A] = 0 \iff Q[A] = 0 \) for each \( A \in \mathcal{F} \). Show that if a sequence \( X_n \) of random variables converges to zero in probability under \( P \) then it does so under \( Q \) too.

\( L_p \) Convergence

7. Find a sequence of RVs \( \{X_n\} \subset L_2 \) which converge in \( L_1 \) but not in \( L_2 \).

8. Let \((\Omega, \mathcal{F}, P) := ((0,1], \mathcal{B}, \lambda)\) be the unit interval with Borel sets and Lebesgue measure and define \( X_n(\omega) := \omega^n \) for \( n \in \mathbb{N}, \omega \in \Omega \). For what \( p \in [1, \infty] \), does the sequence \( \{X_n\} \) converge in \( L_p \)? To what limit? Explain your answer.

9. Verify Hölder’s inequality for \( p = 1, q = \infty \) and all random variables \( X, Y \):
\[
E[XY] \leq \|X\|_1 \|Y\|_\infty
\]
where \( \|Y\|_\infty := \sup \{c < \infty : P(|Y| > c) > 0\} \).

10. Verify Minkowski’s inequality for \( p = \infty \) and all random variables \( X, Y \):
\[
\|X + Y\|_\infty \leq \|X\|_\infty + \|Y\|_\infty
\]
\(^1\)Many other metrics would work too— like \( E[|X - Y| \wedge 1] \) or \( E[|X - Y|/(1 + |X - Y|)] \).
Uniform Integrability (UI)

11. Fix $p > 0$ and set $X_n := n^p 1_{\{0 < \omega \leq 1/n\}}$ on $(\Omega, \mathcal{F}, P)$ with $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $P = \lambda$. Show explicitly that $\{X_n\}$ is UI for $p < 1$ and not for $p \geq 1$, by verifying that $E[X_n 1_{\{X_n > t\}}]$ converges to zero uniformly as $t \to \infty$ for $p < 1$ and not for $p \geq 1$.

12. Let $\{X_n\}$ be an iid sequence of $L_1$ random variables and set $S_n := \sum_{i=1}^{n} X_i$. Show that the sequence of random variables $\{\bar{X}_n\}$ defined by $\bar{X}_n := S_n/n$ is UI.

13. Let $\{X_n\}$ be iid and $L_1$. Show\(^2\):

$$P\left( \lim_{n \to \infty} \frac{X_n}{n} = 0 \right) = 1.$$  

14. If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

15. Let $\phi(x) \geq 0$ be a nonnegative function which grows faster than $x$ at infinity, i.e., $\phi(x)/x \to \infty$ as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$E(\phi(|Z|)) \leq B.$$  

Show that $\mathcal{C}$ is UI. In particular, any collection of random variables that is bounded uniformly in $L_p$ for some $p > 1$ is also UI.

---

\(^2\)Although $\{X_i\}$ are UI, that won’t be a factor in solving this problem. Is independence needed?