This is a closed-book examination, so please do not refer to your notes, the text, or to any other books. You may use a single sheet of your own notes, if you wish, but you may not share materials. A normal distribution table and a blank worksheet are attached to the exam. If you don’t understand something in one of the questions feel free to ask me, but please do not talk to each other. Please acknowledge the Duke Honor Code:

I have neither given nor received aid on this examination: ____________.

You must show some work to get credit—unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give you credit for solutions I do not understand. You should spend about 15–20 minutes on each problem. Good luck.

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1. Events $A$ and $B$ each have probability $1/2$, while their union has probability $P[A \cup B] = 5/8$. Find:

   a. The conditional probability of $A$, if $B$ occurs:

   $$P[A|B] = \underline{\hspace{3cm}}$$

   b. The probability that $A$ and $B$ both occur:

   $$P[A \cap B] = \underline{\hspace{3cm}}$$

   c. The probability of the event $C$ that $A$ occur or $B$ occurs but not both:

   $$P[C] = \underline{\hspace{3cm}}$$
2. You choose a number $n$. I throw $n$ identical fair gold coins into the air and if they all fall Heads, you get to keep them all; but if any falls Tails you get nothing. How many coins would you like me to throw? Why?

OPTIONAL (for extra credit): You choose a number $n$. I roll $n$ identical gold dice and if none shows an Ace, you get to keep them all; but if any ace appears you get nothing. How many dice would you like me to roll? Why?
STA 104
MTH 135 Probability Final Examination

3. The three doors on the Monty Hall show open to show a goat; another goat; and a new car. If a goat cost $1,000 and a car cost $10,000, and if each contestant chooses a door at random (probability 1/3 each) and sticks with it (no “switching”),

a. What is the expected value of one contestant’s prize, X?

\[ E[X] = \] 

b. What is the variance?

\[ V[X] = \] 

c. Let \( X_n \) be the prize won by the \( n^{th} \) contestant and let \( S_{100} = X_1 + \ldots + X_{100} \) be the sum. In 100 shows, how much money should Monty Hall’s producers expect to spend on prizes? What would be the variance? Show your work.

\[ E[S_{100}] = \] \[ V[S_{100}] = \] 

d. Approximately what is the probability that the prize total will exceed $350,000? Show your work.

\[ P[S_{100} \geq 350,000] \approx \]
4. Every time Chris and Pat run the 100 meter dash they do it in different times; Chris’s times (in seconds) follow the exponential distribution $f(t) = \lambda e^{-\lambda t}$ (for $t > 0$) with $\lambda = 0.10$, while Pat’s times follow the same distribution with $\lambda = 0.08$.

   a. What is the probability that Chris beats Pat in a 100m dash?

   $$\Pr[C < P] = \text{______________}$$

   b. What is the probability that Chris beats the world record, 9.88s?

   $$\Pr[C < 9.88] = \text{______________}$$
5. Throw a fair die three times, and call the numbers thrown on the three trials \(X_1\), \(X_2\), and \(X_3\).

   a. What is the probability that the middle number is (strictly) larger than the other two?

   \[
P[X_2 > \max(X_1, X_3)] = \underline{\text{__________}}
   \]

   b. What is the expected product?

   \[
   \mathbb{E}[X_1 \cdot X_2 \cdot X_3] = \underline{\text{__________}}
   \]
STA 104
MTH 135     Probability Final Examination
6. The cappuccino machine at the Cambridge Inn can be adjusted to dispense \textit{about} \( \mu \) ounces (oz) each time, for any \( 0 < \mu < 10 \), but the actual amount dispensed is variable with a standard deviation of 0.1 oz. If the amounts dispensed follow a normal probability distribution, how should \( \mu \) be set to ensure that only one 6 oz cup in a hundred overflows, on average?
probability Final Examination

Pat and Chris are equally likely to shoot an arrow over our shoulder toward the target, but they are not equally talented archers. Pat’s arrows hit the target at points \((R_p, \theta_p)\) (in polar coordinates) with \(\theta_p\) uniformly distributed in the interval \([0, 2\pi)\) and \(R_p\) satisfying 
\[
P[R_p > r] = e^{-4r^2} \text{ for } r > 0,
\]
indep-\(\text{ependently; for Chris’s arrows the distribution of } \theta \text{ is the same but } P[R_c > r] = e^{-r^2}.

a. What is the probability density function for \(R_p\), Pat’s distance from the center?

\[
f_p(r) = \text{______________}
\]

b. If both archers shoot, what is the probability that even the worse shot is inside the \(r = 1/2\) circle?

\[
P[\max(R_p, R_c) < 1/2] = \text{______________}
\]

c. SWOOSH! Suddenly an arrow flies over our shoulder and hits the bull’s eye, \(R \leq 0.10\). What is the probability that it was Pat who shot the arrow, if they were equally likely to shoot?

\[
P[\text{Pat}] = \text{______________}
\]
8. Pat is still an archer whose arrows hit polar-coordinate points \((r, \theta)\) with \(\theta\) uniformly distributed in the interval \([0, 2\pi]\) and \(R\) satisfying \(P[R_p > r] = e^{-4r^2}\) for \(r > 0\), with \(R\) and \(\theta\) independent. I’m not sure where Chris went.

a. What is the JOINT probability density function for \(R\) and \(\theta\)?

\[
f(r, \theta) = 
\]

b. Let \(X = r \cos(\theta)\) and \(Y = r \sin(\theta)\) be the Cartesian coordinates of Pat’s point. It’s possible to show that the joint density function for \(X\) and \(Y\) is

\[
f_{xy}(x, y) = \frac{4}{\pi} e^{-4(x^2 + y^2)}\]

for \(-\infty < x, y < \infty\) (note that \(R\) is not restricted to the unit interval, and \(X\) and \(Y\) need not be less than one in absolute value). Are \(X\) and \(Y\) independent? Why? (NOTE: No integration is needed)

Yes  No  Why?
Pat is still an archer whose arrows hit points at distance $R$ from the center of the target satisfying $P[R_p > r] = e^{-4r^2}$ for $r > 0$. The target has radius one. Chris is on the way to the hospital with an arrow wound.

a. What is the probability of the event $T = \{R \leq 1\}$ that Pat hits the target at all?

$$P[T] = \text{_______________}$$

b. What is the (exact or approximate) probability that Pat misses the target exactly once in 100 tries?

$$P[\text{Exactly One Miss}] = \text{_______________}$$
10. A US deck of playing cards contains 52 cards of four “suits” (♠, ♦, ♣, ♠) and thirteen “ranks” (2,3,4,5,6,7,8,9,10,J,Q,K,A). Imagine dealing cards (just once) from a well-shuffled deck, without replacement. Let $H_1$ be the event “Heart (♥) on 1st card”, $H_2$ be “Heart on 2nd card”, etc.; similarly let $D_n$ be the event “Diamond (♦) on n<sup>th</sup> card” and $A_n$ be the event “Ace (A) on n<sup>th</sup> card”.

a. Give the indicated probabilities:

\[ P[H_1] = \quad \quad \quad P[D_1] = \quad \quad \quad P[A_1] = \]

b. Give the indicated probabilities:

\[ P[H_2] = \quad \quad \quad P[H_1 \cap H_2] = \quad \quad \quad P[H_1|H_2] \]

c. Give the indicated probabilities:

\[ P[H_1 \cap D_1] = \quad \quad \quad P[H_1 \cap A_1] = \quad \quad \quad P[A_1 \cap H_2] \]

d. Which pairs of events are independent? Circle them.

\{H_1, D_1\} \quad \{H_1, A_1\} \quad \{A_1, H_2\} \quad \{H_1, H_2\} \quad \{H_1, D_2\}
Normal Distribution Table

\[
\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt
\]

Table 5.1 Area \( \Phi(x) \) under the Standard Normal Curve to the left of \( x \).

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STA 104
MTH 135
Extra worksheet, if needed

Name: ____________________________

Probability Second Test