Practice Final Exam: Statistical Inference

Wed Apr 24

1. Let $X_j \sim \mathsf{Ex}(\theta)$ be independent exponentially-distributed random variables, each with density function

$$f(x \mid \theta) = \theta\, e^{-\theta\, x}, \qquad x > 0, \tag{1}$$

and answer the following questions.

   a. Find the posterior distribution for θ and the posterior mean $\mathsf{E}[\theta \mid X]$ for n observations $x_1, \ldots, x_n$, using a Gamma prior distribution $\theta \sim \mathsf{Ga}(\alpha, \lambda)$.

   b. Find the Jeffreys' prior $\pi_J(d\theta)$, the posterior distribution for θ, and the posterior mean $\mathsf{E}_J[\theta \mid X]$.

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2. With the same data and model as above, and with a single observation ($n = 1$) of $X_1 = x = 0.10$,

   a. Find an exact one-sided 90% confidence interval $I = [0, R_x]$ so that $\mathsf{P}[\theta \in I \mid \theta] = 0.90$ for every $\theta > 0$.

   b. Perform a significance test of the hypothesis $H_0 : \theta \leq 1$ against the alternative $H_1 : \theta > 1$. Find the P-value and report whether or not to reject the hypothesis at level $\alpha = .05$.

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3. Once again with the same exponential model as above, verify that, for any $n \geq 1$, the statistics $T_1(X) = \sum_{j \leq n} X_j$ and $T_2(X) = \min_{j \leq n} X_j$ have the distributions $T_1 \sim \mathsf{Ga}(n, \theta)$ and $T_2 \sim \mathsf{Ex}(n\theta)$, respectively. Which of these would be a better basis for inference about θ? Why?

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4. Yet again let $X_j \sim \mathsf{Ex}(\theta)$ be independent exponentially-distributed random variables for $j \le n$, and set $\mu = \mathsf{E}[X_j \mid \theta] = 1/\theta$; one *could* write the model as

$$f(x \mid \mu) = \mu^{-1}\, e^{-x/\mu}, \qquad x > 0. \tag{2}$$

Please answer the following questions.

a. For the improper uniform distribution $\pi(d\theta) = d\theta$, find the posterior distribution for θ and the derived posterior distribution for $\mu = 1/\theta$.

b. For the improper uniform distribution $\pi(d\mu) = d\mu$, find the posterior distribution for μ and the derived posterior distribution for $\theta = 1/\mu$. Comment on the relation to 4a) above.

c. Find the Jeffreys prior distribution $\pi_J(d\mu)$ and the Jeffreys posterior distribution for for μ and the derived posterior distribution for $\theta = 1/\mu$. Comment on the relationship to your answer 1b).

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5. Let $X \sim \mathsf{Bi}(n, \theta)$ for $\theta \in \Theta = \{\frac{1}{2}, \frac{2}{3}\}$.

a. Find the posterior odds ratio $\mathsf{P}[\theta = \frac{1}{2} \mid X]/\mathsf{P}[\theta = \frac{2}{3} \mid X]$, as a function of the observed data $X = x$ and the prior odds ratio $\omega \equiv \mathsf{P}[\theta = \frac{1}{2}]/\mathsf{P}[\theta = \frac{2}{3}]$.

b. With prior odds ratio $\omega = 1$, give the posterior probability of the hypothesis $H_0 : \theta = \frac{1}{2}$ on the basis of an observed $x = 8$ with $n = 10$.

c. Find the observed significance level (P-value) for a test of this same hypothesis. Would you accept or reject at level $\alpha = .05$?
