Homework 4

Due: Friday 2:00pm, Feb 28, 2003

0. Turn in the simulation study report from Lab5.

1. 6.52

2. 7.32 (Take a look of example 7.11(c) in the textbook. Remember to use continuity correction in the approximation)

3. The joint probability distribution $p(x, y)$ is given by

$$p(-1,0) = p(0,-1) = p(0,1) = p(1,0) = 1/4.$$ 

Show that the correlation between $x$ and $y$ is zero, but $x$ and $y$ are not independent.  

Hint: To show that $x$ and $y$ are not independent, it is enough to show that there exists a pair $(x_0, y_0)$ such that $p(x_0, y_0) \neq p_x(x_0)p_y(y_0)$.

4. Annie and Alvie have agreed to meet between 5pm and 6pm for dinner at Great Hall. Let $x =$ Annie’s arrival time and $y =$ Alvie’s arrival time. Suppose $x$ and $y$ are independent with each uniformly distributed on the interval $[5,6]$.

(a) What is the joint pdf of $x$ and $y$?

(b) What is the probability that they both arrive between 5:15 and 5:45?

(c) If the first one to arrive will wait only 10 min before leaving to eat elsewhere, what is the probability that they have dinner at Great Hall?  

Hint: The event of interest is $A = \{(x,y) : |x-y| \leq 1/6\}$.

5. Two components of a minicomputer have the following joint pdf for their useful lifetime $x$ and $y$:

$$f(x,y) = \begin{cases} 
xe^{-(1+y)} & x \geq 0 \text{ and } y \geq 0 \\
0 & \text{otherwise} 
\end{cases}$$

(a) What is the probability that the lifetime $x$ of the first component exceeds 3?

(b) Are $x$ and $y$ independent? Explain.

Hint:  (1) First calculate the marginal pdf’s of random variables $x$ and $y$, then check whether the joint pdf $f(x,y)$ is equal to the product of two marginals.

(2) When you calculate the marginal density of $y$, you will encounter a integration like this form: 

$$\int_0^\infty x^{a-1}e^{-x/\beta}dx,$$

which is equal to 

$$\beta^a\Gamma(a) \int_0^\infty \frac{1}{\beta^a\Gamma(a)} x^{a-1}e^{-x/\beta}dx = \beta^a\Gamma(a).$$

Here we use the fact that the function inside the integral is a Gamma$(\alpha, \beta$) density function, therefore the integral is equal 1.

(3) $\Gamma(n+1) = n!$, therefore, $\Gamma(2) = 1$.

6. (Let’s redo the following problem from the exam.) Consider a system consisting of two components as in the figure below.
The system works if *either 1 or 2 works*. Suppose the lifetime of each component, measured in hours, is a random variable $x_i$, $i = 1, 2$, with density function

$$f_{x_i}(x) = \begin{cases} \frac{1}{\beta_i}e^{-x/\beta_i}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

where $\beta_1 = 2$ hours and $\beta_2 = 1.5$ hours and assume that the components are independent.

(a) Let $t$ = the lifetime of the system. Find the density function of the random variable $t$.

(b) Calculate the probability that the system will operate for more than 6 hours, using the density function calculated at (a).

*Hint:* Use the cumulative distribution function method. First calculate the probability of the event \{t \leq t_0\} and then differentiate.