Solution for HW5

STA113, ISDS

Total 14 points

1, (4 points)

- a. To find the moment estimator of p, you just need to equal the moments of population to the sample moments. Note that the first moment is enough because we just have one parameter. $\hat{p} = y/n$.

- b. To check whether an estimator unbiased, we need to check if its expectation is equal to the estimated parameter. In this problem, we need to check if $E(\hat{p}) = p$. It is straightforward to see that the moment estimator is unbiased.

- c. The likelihood function, $L(p; y) = p^y(1-p)^{n-y}$. Maximizing $l(p; y) = \log L(p; y)$ can reach the MLE;

$$\begin{aligned} l(p; y) &= y \log(p) + (n-y) \log(1-p) \\ \frac{d_{l(p;y)}}{d_p} &= \frac{y}{p} - \frac{n-y}{1-p} \\ &= 0 \\ \hat{p} &= y/n \end{aligned}$$

  (Note that if $X \sim Bernoulli(p)$, then the mass function of X, say $f(x)$, can be expressed as $f(x) = p^x(1-p)^{1-x}$).

- d. Yes, the MLE is also unbiased. As we can see in this problem, the MLE and the moment estimator are the same. But it is not always the case.

2, (2 points)

- a.

$$\begin{aligned} L(\lambda; \overrightarrow{Y}) &= \prod p(y_i) \\ &\propto e^{-n\lambda}\lambda^{\sum(y_i)} \\ &\doteq f(\lambda) \end{aligned}$$

  To get the MLE, we just need to maximize $f(\lambda)$.

$$\begin{aligned} \log f(\lambda) &= -n\lambda + (\sum(y_i))\log(\lambda) \\ \frac{d(f(\lambda))}{d_\lambda} &= -n + (\sum(y_i))/\lambda \\ &= 0 \\ \hat{\lambda} &= (\sum y_i)/n \end{aligned}$$

  Note that in the previous work, we drop the items $y!$ because they are constants with regard to λ.

- b. $E(\hat{\lambda}) = E(y_i) = \lambda$. So the MLE is unbiased.

3, 3 points

- a. $P(\omega < \omega_0) = \prod_{i=1}^n P(y_i < \prod_{i=1}^n \omega_0) = \int_0^{\omega_0} 1/\theta dy_i$. Then take derivative with respect to ω_0, hence the density function $n\frac{\omega_0^{n-1}}{\theta^n}$.

- b. $E\hat{\theta} = \frac{n}{n+1}\theta$.

- c. $\hat{\theta} = 2\overline{y}$. We can show that $E\hat{\theta} = \theta$. Another unbiased estimator is given by $(n+1)\theta_{mle}/n$. You can see it is unbiased from part b.

4, (1 points)

$$\begin{aligned} E(\bar{y}) &= \lambda \\ var(\bar{y}) &= var(\sum(y_i))/n^2 \\ &= \lambda/n \end{aligned}$$

By Central Limit Theorem, we have $z = (\bar{y} - \lambda)/\sqrt{\lambda/n}$ is approximately a standard normal. Note that when the sample size is large, by the Large Number Theory, $\bar{y} \approx \lambda$. Thus we can substitute $\bar{y}$ for λ in the denominator, and get an approximate confidence interval as $(\bar{y}-z_{1-\alpha/2}\sqrt{\bar{y}/n}, \bar{y}-z_{\alpha/2}\sqrt{\bar{y}/n})$, where z_α denotes the α percentile of a standard normal distribution.

5, (2 points)

- a. Longer.
- b. No. See the lower bound and upper bound rather than μ as random variables.

6, (2 points)

- a. Notice $\sqrt{n}\frac{\bar{y}-\mu}{\sigma}$ has a distribution of $N(0,1)$. Hence the 95% CI is $[1003, 1025]$.
- b. Solve $\sqrt{n}\frac{6/2}{\sigma} \geqslant 1.96$ for n. We have $n \geqslant 267$. So at least we need the sample size to be 267.