This measure on \( \mathcal{F} \) is the required extension, because by (3.7) it agrees with \( P \) on \( \mathcal{F}_0 \).

**Uniqueness and the \( \pi\)-\( \lambda \) Theorem**

To prove the extension in Theorem 3.1 is unique requires some auxiliary concepts. A class \( \mathcal{P} \) of subsets of \( \Omega \) is a \( \pi\)-system if it is closed under the formation of finite intersections:

\[
(\pi) \quad A, B \in \mathcal{P} \text{ implies } A \cap B \in \mathcal{P}.
\]

A class \( \mathcal{L} \) is a \( \lambda\)-system if it contains \( \Omega \) and is closed under the formation of complements and of finite and countable disjoint unions:

\[
(\lambda_1) \quad \Omega \in \mathcal{L};
(\lambda_2) \quad A \in \mathcal{L} \text{ implies } A^c \in \mathcal{L};
(\lambda_3) \quad A_1, A_2, \ldots, \in \mathcal{L} \text{ and } A_n \cap A_m = \emptyset \text{ for } m \neq n \text{ imply } \bigcup_n A_n \in \mathcal{L}.
\]

Because of the disjointness condition in (\( \lambda_3 \)), the definition of \( \lambda\)-system is weaker (more inclusive) than that of \( \sigma\)-field. In the presence of (\( \lambda_1 \)) and (\( \lambda_2 \)), which imply \( \emptyset \in \mathcal{L} \), the countably infinite case of (\( \lambda_3 \)) implies the finite one.

In the presence of (\( \lambda_1 \)) and (\( \lambda_3 \)), (\( \lambda_2 \)) is equivalent to the condition that \( \mathcal{L} \) is closed under the formation of proper differences:

\[
(\lambda'_2) \quad A, B \in \mathcal{L} \text{ and } A \subset B \text{ imply } B - A \in \mathcal{L}.
\]

Suppose, in fact, that \( \mathcal{L} \) satisfies (\( \lambda_3 \)) and (\( \lambda'_2 \)). If \( A, B \in \mathcal{L} \) and \( A \subset B \), then \( \mathcal{L} \) contains \( B^c \), the disjoint union \( A \cup B^c \), and its complement \( (A \cup B^c)^c = B - A \). Hence (\( \lambda_3 \)). On the other hand, if \( \mathcal{L} \) satisfies (\( \lambda_1 \)) and (\( \lambda'_2 \)), then \( A \in \mathcal{L} \) implies \( A^c = \Omega - A \in \mathcal{L} \). Hence (\( \lambda_3 \)).

Although a \( \sigma\)-field is a \( \lambda\)-system, the reverse is not true (in a four-point space take \( \mathcal{L} \) to consist of \( \emptyset, \Omega \), and the six two-point sets). But the connection is close:

**Lemma 6.** A class that is both a \( \pi\)-system and a \( \lambda\)-system is a \( \sigma\)-field.

**Proof.** The class contains \( \Omega \) by (\( \lambda_1 \)) and is closed under the formation of complements and finite intersections by (\( \lambda_2 \)) and (\( \pi \)). It is therefore a field. It is a \( \sigma\)-field because if it contains sets \( A_n \), then it also contains the disjoint sets \( B_n = A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c \) and by (\( \lambda_3 \)) contains \( \bigcup_n A_n = \bigcup_n B_n \).
Many uniqueness arguments depend on Dynkin's $\pi$-$\lambda$ theorem:

**Theorem 3.2.** If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system, then $\mathcal{P} \subseteq \mathcal{L}$ implies $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.

**Proof.** Let $\mathcal{L}_0$ be the $\lambda$-system generated by $\mathcal{P}$—that is, the intersection of all $\lambda$-systems containing $\mathcal{P}$. It is a $\lambda$-system, it contains $\mathcal{P}$, and it is contained in every $\lambda$-system that contains $\mathcal{P}$ (see the construction of generated $\sigma$-fields, p. 21). Thus $\mathcal{P} \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$. If it can be shown that $\mathcal{L}_0$ is also a $\pi$-system, then it will follow by Lemma 6 that it is a $\sigma$-field. From the minimality of $\sigma(\mathcal{P})$ it will then follow that $\sigma(\mathcal{P}) \subseteq \mathcal{L}_0$, so that $\mathcal{P} \subseteq \sigma(\mathcal{P}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$. Therefore, it suffices to show that $\mathcal{L}_0$ is a $\pi$-system.

For each $A$, let $\mathcal{L}_A$ be the class of sets $B$ such that $A \cap B \in \mathcal{L}_0$. If $A$ is assumed to lie in $\mathcal{P}$, or even if $A$ is merely assumed to lie in $\mathcal{L}_0$, then $\mathcal{L}_A$ is a $\lambda$-system: Since $A \cap \Omega = A \in \mathcal{L}_0$ by the assumption, $\mathcal{L}_A$ satisfies (\lambda_1). If $B_1, B_2 \in \mathcal{L}_A$ and $B_1 \subseteq B_2$, then the $\lambda$-system $\mathcal{L}_0$ contains $A \cap B_1$ and $A \cap B_2$ and hence contains the proper difference $(A \cap B_2) - (A \cap B_1) = A \cap (B_2 - B_1)$, so that $\mathcal{L}_A$ contains $B_2 - B_1$: $\mathcal{L}_A$ satisfies (\lambda_2). If $B_n$ are disjoint $\mathcal{L}_A$-sets, then $\mathcal{L}_0$ contains the disjoint sets $A \cap B_n$ and hence contains their union $A \cap (\bigcup_n B_n)$: $\mathcal{L}_A$ satisfies (\lambda_3).

If $A \in \mathcal{P}$ and $B \in \mathcal{P}$, then (\mathcal{P} is a $\pi$-system) $A \cap B \in \mathcal{P} \subseteq \mathcal{L}_0$, or $B \in \mathcal{L}_A$. Thus $A \in \mathcal{P}$ implies $\mathcal{P} \subseteq \mathcal{L}_A$, and since $\mathcal{L}_A$ is a $\lambda$-system, minimality gives $\mathcal{L}_0 \subseteq \mathcal{L}_A$.

Thus $A \in \mathcal{P}$ implies $\mathcal{L}_0 \subseteq \mathcal{L}_A$, or, to put it another way, $A \in \mathcal{P}$ and $B \in \mathcal{L}_0$ together imply that $B \in \mathcal{L}_A$ and hence $A \in \mathcal{L}_B$. (The key to the proof is that $B \in \mathcal{L}_A$ if and only if $A \in \mathcal{L}_B$.) This last implication means that $B \in \mathcal{L}_0$ implies $\mathcal{P} \subseteq \mathcal{L}_B$. Since $\mathcal{L}_B$ is a $\lambda$-system, it follows by minimality once again that $B \in \mathcal{L}_0$ implies $\mathcal{L}_0 \subseteq \mathcal{L}_B$. Finally, $B \in \mathcal{L}_0$ and $C \in \mathcal{L}_0$ together imply $C \in \mathcal{L}_B$, or $B \cap C \in \mathcal{L}_0$. Therefore, $\mathcal{L}_0$ is indeed a $\pi$-system.

Since a field is certainly a $\pi$-system, the uniqueness asserted in Theorem 3.1 is a consequence of this result:

**Theorem 3.3.** Suppose that $P_1$ and $P_2$ are probability measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a $\pi$-system. If $P_1$ and $P_2$ agree on $\mathcal{P}$, then they agree on $\sigma(\mathcal{P})$.

**Proof.** Let $\mathcal{L}$ be the class of sets $A$ in $\sigma(\mathcal{P})$ such that $P_1(A) = P_2(A)$. Clearly $\Omega \in \mathcal{L}$. If $A \in \mathcal{L}$, then $P_1(A^c) = 1 - P_1(A) = 1 - P_2(A) = P_2(A^c)$, and hence $A^c \in \mathcal{L}$. If $A_n$ are disjoint sets in $\mathcal{L}$, then $P_1(\bigcup_n A_n) = \sum_n P_1(A_n) = \sum_n P_2(A_n) = P_2(\bigcup_n A_n)$, and hence $\bigcup_n A_n \in \mathcal{L}$. Therefore $\mathcal{L}$ is a $\lambda$-system. Since by hypothesis $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{P}$ is a $\pi$-system, the $\pi$-$\lambda$ theorem gives $\sigma(\mathcal{P}) \subseteq \mathcal{L}$, as required.
Note that the \( \pi\lambda \) theorem and the concept of \( \lambda \)-system are exactly what are needed to make this proof work: The essential property of probability measures is countable additivity, and this is a condition on countable disjoint unions, the only kind involved in the requirement \( \lambda_\lambda \) in the definition of \( \lambda \)-system. In this, as in many applications of the \( \pi\lambda \) theorem, \( \mathcal{L} \subset \sigma (\mathcal{P}) \) and therefore \( \sigma (\mathcal{P}) = \mathcal{L} \), even though the relation \( \sigma (\mathcal{P}) \subset \mathcal{L} \) itself suffices for the conclusion of the theorem.

Monotone Classes

A class \( \mathcal{M} \) of subsets of \( \Omega \) is monotone if it is closed under the formation of monotone unions and intersections:

(i) \( A_1, A_2, \ldots \in \mathcal{M} \) and \( A_n \uparrow A \) imply \( A \in \mathcal{M} \);
(ii) \( A_1, A_2, \ldots \in \mathcal{M} \) and \( A_n \downarrow A \) imply \( A \in \mathcal{M} \).

Halmos's monotone class theorem is a close relative of the \( \pi\lambda \) theorem but will be less frequently used in this book.

**Theorem 3.4.** If \( \mathcal{F}_0 \) is a field and \( \mathcal{M} \) is a monotone class, then \( \mathcal{F}_0 \subset \mathcal{M} \) implies \( \sigma (\mathcal{F}_0) \subset \mathcal{M} \).

**Proof.** Let \( m (\mathcal{F}_0) \) be the minimal monotone class over \( \mathcal{F}_0 \)—the intersection of all monotone classes containing \( \mathcal{F}_0 \). It is enough to prove \( \sigma (\mathcal{F}_0) \subset m (\mathcal{F}_0) \); this will follow if \( m (\mathcal{F}_0) \) is shown to be a field, because a monotone field is a \( \sigma \)-field.

Consider the class \( \mathcal{S} = \{ A : A \in m (\mathcal{F}_0) \} \). Since \( m (\mathcal{F}_0) \) is monotone, so is \( \mathcal{S} \). Since \( \mathcal{F}_0 \) is a field, \( \mathcal{F}_0 \subset \mathcal{S} \), and so \( m (\mathcal{F}_0) \subset \mathcal{S} \). Hence \( m (\mathcal{F}_0) \) is closed under complementation.

Define \( \mathcal{S}_1 \) as the class of \( A \) such that \( A \cup B \in m (\mathcal{F}_0) \) for all \( B \in \mathcal{F}_0 \). Then \( \mathcal{S}_1 \) is a monotone class and \( \mathcal{F}_0 \subset \mathcal{S}_1 \); from the minimality of \( m (\mathcal{F}_0) \) follows \( m (\mathcal{F}_0) \subset \mathcal{S}_1 \).

Define \( \mathcal{S}_2 \) as the class of \( B \) such that \( A \cup B \in m (\mathcal{F}_0) \) for all \( A \in m (\mathcal{F}_0) \). Then \( \mathcal{S}_2 \) is a monotone class. Now from \( m (\mathcal{F}_0) \subset \mathcal{S}_1 \) it follows that \( A \in m (\mathcal{F}_0) \) and \( B \in \mathcal{F}_0 \) together imply that \( A \cup B \in m (\mathcal{F}_0) \); in other words, \( B \in \mathcal{F}_0 \) implies that \( B \in \mathcal{S}_2 \), thus \( \mathcal{F}_0 \subset \mathcal{S}_2 \); by minimality, \( m (\mathcal{F}_0) \subset \mathcal{S}_2 \), and hence \( A, B \in m (\mathcal{F}_0) \) implies that \( A \cup B \in m (\mathcal{F}_0) \). \( \blacksquare \)

Lebesgue Measure on the Unit Interval

Consider once again the unit interval \( (0, 1] \) together with the field \( \mathcal{B}_0 \) of finite disjoint unions of subintervals (Example 2.2) and the \( \sigma \)-field \( \mathcal{B} = \sigma (\mathcal{B}_0) \) of Borel sets in \( (0, 1] \). According to Theorem 2.2, (2.12) defines a probability measure \( \lambda \) on \( \mathcal{B}_0 \). By Theorem 3.1, \( \lambda \) extends to \( \mathcal{B} \), the extended \( \lambda \) being Lebesgue measure. The probability space \( ((0, 1], \mathcal{B}, \lambda) \) will be the basis for much of the probability theory in the remaining sections of this chapter. A few geometric properties of \( \lambda \) will be considered here. Since the intervals in \( (0, 1] \) form a \( \pi \)-system generating \( \mathcal{B} \), \( \lambda \) is the only probability measure on \( \mathcal{B} \) that assigns to each interval its length as its measure.