Midterm Examination

STA 205: Probability and Measure Theory

Thursday, 2003 Feb 27, 12:40-1:55 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. You may use a calculator but not a laptop, pda, etc. If a question seems ambiguous or confusing please ask me—don’t guess, and don’t discuss exam questions with others.

Unless a problem states otherwise, you must show your work to get partial credit. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

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Print Name: __________________________
Problem 1: Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the probability space \(\Omega = \{a, b, c, d\}\) with \(\sigma\)-field \(\mathcal{F} = 2^\Omega\) and probability assignment given by \(\mathbb{P}(\{\omega\}) = \{0.2, 0.1, 0.3, 0.4\}\) for \(\omega = \{a, b, c, d\}\), respectively.

a. (5) List all possible values \(0 \leq p \leq 1\) for which it is possible to construct a random variable \(X_p\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with the Bernoulli distribution with \(\mathbb{P}[X_p = 1] = p = 1 - \mathbb{P}[X = 0]\). If possible, give \(X_p\) explicitly for \(p = 1/2\); if not, tell why not.

\[ p \in \{ \quad \} \quad \quad X_{1/2}(\omega) = \quad \]  

b. (5) Is it possible to construct a random variable \(Y\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with exactly three possible outcomes, all equally likely? If so, do it (explicitly); if not, tell why.

\[ \bigcirc \text{Yes} \quad \bigcirc \text{No} \quad Y(\omega) = \quad \]
Problem 1 (cont’d): Recall $\Omega = \{a, b, c, d\}$ and $\mathbb{P}(\{\omega\}) = \{0.2, 0.1, 0.3, 0.4\}$ for $\omega = \{a, b, c, d\}$.

c. (5) Give (explicitly) the $\sigma$-field $\mathcal{G} = \sigma(W, Z)$ generated by the two random variables $W(\omega) = 1_{\{c\}}(\omega)$ and $Z(\omega) = 1_{\{a,b\}}(\omega)$ (where $1_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise).

$$\mathcal{G} = \{\emptyset, \{c\}, \{a,b\}, \Omega\}$$

d. (5) Is your $\sigma$-field $\mathcal{G}$ (from c. above) complete for $\mathbb{P}$?
   - $\bigcirc$ Yes  $\bigcirc$ No  Why?
Problem 2: A real-valued function $\phi$ on a convex set $S$ is convex if the line segment connecting any two points $[x, \phi(x)]$ and $[y, \phi(y)]$ on the graph $\{(s, \phi(s)) : s \in S\}$ of $\phi$ lies above the graph:

$$\forall x, y \in S, \; 0 < t < 1, \; \phi(x + t(y - x)) \leq \phi(x) + t[\phi(y) - \phi(x)]$$

If $\phi$ is continuous it is enough to check this for the midpoint, i.e., a continuous function $\phi$ is convex if

$$\forall x, y \in S, \; \phi\left(\frac{x + y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2}.$$  

A twice differentiable function $\phi$ on an interval is convex if $\phi''(x) \geq 0$ everywhere. Let $\phi, \psi$ both be continuous and convex on an interval $S \subset \mathbb{R}$. You may assume differentiability if you wish. Choose Yes or No (3 pts) and give support each answer with a brief argument or a sketch of the graphs of $\phi$ and $\psi$ (2pts):

a. (5) Is the sum $(\phi + \psi)$ convex? ○ Yes  ○ No

b. (5) Is the difference $(\phi - \psi)$ convex? ○ Yes  ○ No

c. (5) Is the maximum $(\phi \vee \psi)$ convex? ○ Yes  ○ No

d. (5) Is the minimum $(\phi \wedge \psi)$ convex? ○ Yes  ○ No
Problem 3: Some economists use the Pareto distribution to model incomes; a simple version of this distribution has

\[ P[X > x] = x^{-\alpha}, \quad x > 1 \]

for some parameter \( \alpha > 0 \). Let \( X \) have this Pareto distribution.

a. (5) Give the probability density function \( f(x) \) for \( X \). Be careful about the support, i.e., the set where \( f(x) > 0 \).

\[ f(x) = \quad \text{__________} \]

b. (15) For which real numbers \( p \in (0, \infty) \) is \( X \in L_p \)? Note this may depends upon \( \alpha \). Evaluate \( E[X^p] \).

\[ X \in L_p \text{ if } p \in \{ \quad \text{________________________} \} \]

\[ E[X^p] = \quad \text{__________} \]
Problem 4: Let \((\Omega, \mathcal{F}, P)\) be the unit interval \(\Omega = (0, 1]\) with Lebesgue measure \(P = \lambda\); for each number \(\alpha \in \mathbb{R} = (-\infty, \infty)\), consider the sequence of random variables (for \(n \in \mathbb{N} = \{1, 2, \ldots\}\))

\[
X_n(\omega) \equiv n^\alpha 1_{(0,1/n]}(\omega) = \begin{cases} 
n^\alpha & 0 < \omega \leq 1/n \\
0 & \text{otherwise.}
\end{cases}
\]

Answer each question below and show why your answer is correct.

a. (4) For each \(\alpha \in \mathbb{R}\) compute the probability of \(\{\omega : X_n(\omega) \to 0\}\):

\[
P[X_n \to 0] = \underline{\text{___________}}
\]

b. (8) For each \(\alpha \in \mathbb{R}\) and \(p > 0\) compute \(E[|X_n|^p]\). For which \(\alpha \in \mathbb{R}\) and \(p > 0\) does \(X_n \to 0\) in \(L_p\)?

\[
E[|X_n|^p] = \underline{\text{___________}} \quad E[|X_n|^p] \to 0 \text{ if } p \in \underline{\text{___________}}
\]

c. (8) For which \(\alpha \in \mathbb{R}\) (if any) is \(\{X_n : n \in \mathbb{N}\}\) “dominated” by some \(Y \in L_1((\Omega, \mathcal{F}, P))\) satisfying \(|X_n| \leq Y\)? Find such a \(Y\) if possible.

\[
\alpha \in \{ \underline{\text{______________}} \} \quad Y = \underline{\text{______________}}
\]
Problem 5: Let $X$ be a simple random variable (with finitely-many values) and $Y$ a continuously-distributed random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathbb{P} = \lambda$. In each case give an explicit non-trivial example of $X$ and $Y$ if possible; if impossible, say why. Do not prove anything!

a. (4) Is it possible that $\sigma(X) \subseteq \sigma(Y)$?  ○ Yes  ○ No

b. (4) Is it possible that $\sigma(Y) \subseteq \sigma(X)$?  ○ Yes  ○ No

c. (4) Is it possible that $\sigma(X) \perp \sigma(Y)$?  ○ Yes  ○ No

d. (4) Is it possible that $\sigma(X) = \mathcal{F}$?  ○ Yes  ○ No

e. (4) Is it possible that $\sigma(Y) = \mathcal{F}$?  ○ Yes  ○ No