Midterm Examination #1

STA 205: Probability and Measure Theory

Thursday, 2004 Feb 16, 2:20-3:35 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. You may use a calculator but not a laptop, pda, etc. If a question seems ambiguous or confusing please ask Jason—don’t guess, and don’t discuss exam questions with others.

Unless a problem states otherwise, you must show your work to get partial credit. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

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Problem 1: Let $(\Omega, \mathcal{F}, P)$ be the probability space $\Omega = \{a, b, c, d\}$ with just four points and define a collection of sets by
\[
\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\}
\]
a. (5) Is $\mathcal{F}$ a field? If yes, state and illustrate what conditions need to be verified (you don’t have to verify these conditions for every possible combination of sets); if no, give a counter-example.
Circle one:  Yes  No  Reasoning:

b. (5) Give a real-valued rand. vble. $X$ that generates $\mathcal{F} = \sigma(X)$:
$X(a) = \underline{\hspace{2cm}}$  $X(b) = \underline{\hspace{2cm}}$  $X(c) = \underline{\hspace{2cm}}$  $X(d) = \underline{\hspace{2cm}}$

c. (5) Find the expectation of your random variable $X$ from b. above, for the uniform probability measure assigning probability $1/4$ to each point of $\Omega$. Show your work.
\[E[X] = \underline{\hspace{2cm}}\]

d. (5) Is the field $\mathcal{F}$ complete for this probability assignment?
Circle one:  Yes  No  Reasoning:
Problem 2: Consider two independent fair dice, each of which shows the numbers \( \{1, 2, 3, 4, 5, 6\} \) with equal probabilities; one is red, and one green. Let \( S = R + G \) be their sum.

a. (10) Write \( R \) and \( G \) as random variables on some probability space \((\Omega, \mathcal{F}, P)\) (you must specify \( \Omega \) and \( \mathcal{F} \); describe how to calculate \( P(A) \) for each \( A \in \mathcal{F} \); and give \( R(\omega) \), \( G(\omega) \) for each \( \omega \in \Omega \):

\[
\Omega = \\
\mathcal{F} = \\
P(A) = \\
R(\omega) = \\
G(\omega) =
\]

b. (10) Give non-trivial examples of a set \( A \in \sigma(R) \) in the \( \sigma \)-field generated by the red die \( R \) and one \( B \in \sigma(S) \) in the \( \sigma \)-field generated by the sum \( S \equiv R + G \), and give their probabilities:

\[ A = \]

\[
P[A] = \underline{\quad} \\
B = \]

\[
P[B] = \underline{\quad}
\]
Problem 3:
Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of random variables, all defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The event that the infimum (greatest lower bound) is at least \( \alpha \in \mathbb{R} \) can be expressed \( \cap_{n=1}^{\infty} [X_n \geq \alpha] \).

a. (8) Write the event that the supremum of \( \{X_n\}_{n \in \mathbb{N}} \) is less than or equal to \( \beta \in \mathbb{R} \), using intersections, unions, etc. to combine events of the form \( [X_n \leq x] \), \( [X_n \geq y] \), etc.:

\[
\sup_{n < \infty} X_n \leq \beta = \text{_______________________________}
\]

b. (8) Write the event that only finitely-many \( X_n \)'s exceed \( \beta \)
(Hint: Think about \( \{X_n : n \geq m\} \) for \( m \in \mathbb{N} \)):

\[
\limsup_{n \to \infty} X_n \leq \beta = \text{_______________________________}
\]

c. (4) Why does this show that, if a sequence of \( \mathcal{F} \)-measurable random variables \( X_n(\omega) \) converges to some value \( X(\omega) \) at every \( \omega \in \Omega \), then the limit \( X \equiv \lim_{n \to \infty} X_n \) is also \( \mathcal{F} \)-measurable?
Problem 4: Let $Y$ be a random variable taking values $0 < Y \leq 1$ and set $X \equiv Y^2$, $Z \equiv \sqrt{Y}$.

a. (10) What does Jensen’s inequality tell you about the relations between the values of the expectations $a = E[X]$, $b = E[Y]$, and $c = E[Z]$? Tell what convex function(s) are you using, and how.

b. (10) If $Y$ is uniformly distributed on $(0, 1]$, compute those three expectations explicitly and verify Jensen’s conclusions:

$$a = E[X] = \underline{\quad} \quad b = E[Y] = \underline{\quad} \quad c = E[Z] = \underline{\quad}$$
Problem 5: Let \((\Omega, \mathcal{F}, P)\) be the open unit interval \(\Omega = (0, 1)\) with the Borel sets \(\mathcal{F} = \mathcal{B}^1\) and Lebesgue measure \(P = \lambda\). Set \(X_n(\omega) = n 1_{(0,1/n^2]}(\omega)\) for each integer \(n \in \mathbb{N}\).

a. (4) Does \(X_n(\omega)\) converge at each point \(\omega \in \Omega\)? If so, find \(\lim_{n \to \infty} X_n(\omega)\); if not, tell why.

Circle one:  Yes  No  \(X(\omega) = \) _______________

Reasoning:

b. (4) Does \(\int_\Omega |X_n - X| dP \to 0\)?  Y  N  Show why...

c. (4) Does \(\int_\Omega |X_n - X|^2 dP \to 0\)?  Y  N  Show why...

d. (4) Is \(\{X_n\}\) uniformly bounded by a positive integrable RV \(Y\)?
If so, find a suitable \(Y\); if not, explain.  Y  N

e. (4) Is \(\{X_n^2\}\) uniformly bounded by a positive integrable RV \(Y\)?
If so, find a suitable \(Y\); if not, explain.  Y  N
Another Blank Worksheet