Midterm Examination #1

STA 205: Probability and Measure Theory

Thursday, 2004 Feb 16, 2:20-3:35 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. You may use a calculator but not a laptop, pda, etc. If a question seems ambiguous or confusing *please* ask Jason— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Print Name: ______________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the probability space $\Omega = \{a, b, c, d\}$ with just four points and define a collection of sets by

$$\mathcal{F} = \big\{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\big\}$$

a. (5) Is $\mathcal{F}$ a field? If *yes*, state and illustrate what conditions need to be verified (you don't have to verify these conditions for every possible combination of sets); if *no*, give a counter-example.

Circle one: Yes No Reasoning:

b. (5) Give a real-valued rand. vble. X that generates $\mathcal{F} = \sigma(X)$:

$X(a) =$ ____ $X(b) =$ ____ $X(c) =$ ____ $X(d) =$ ____

c. (5) Find the expectation of your random variable X from b. above, for the uniform probability measure assigning probability 1/4 to each point of Ω. Show your work.

$\mathsf{E}[X] =$ ______________

d. (5) Is the field $\mathcal{F}$ *complete* for this probability assignment?

Circle one: Yes No Reasoning:

Problem 2: Consider two independent fair dice, each of which shows the numbers $\{1, 2, 3, 4, 5, 6\}$ with equal probabilities; one is red, and one green. Let $S = R + G$ be their sum.

a. (10) Write R and G as random variables on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ (you must specify Ω and $\mathcal{F}$; describe how to calculate $\mathsf{P}(A)$ for each $A \in \mathcal{F}$; and give $R(\omega)$, $G(\omega)$ for each $\omega \in \Omega$):

$\Omega \quad =$

$\mathcal{F} \quad =$

$\mathsf{P}(A) \quad =$

$R(\omega) \quad =$

$G(\omega) \quad =$

b. (10) Give non-trivial examples of a set $A \in \sigma(R)$ in the σ-field generated by the red die R and one $B \in \sigma(S)$ in the σ-field generated by the sum $S \equiv R + G$, and give their probabilities:

$A =$

$\mathsf{P}[A] =$ ______________

$B =$

$\mathsf{P}[B] =$ ______________

Problem 3:

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, all defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The event that the infimum (greatest lower bound) is at least $\alpha \in \mathbb{R}$ can be expressed $\cap_{n=1}^{\infty}[X_n \geq \alpha]$.

a. (8) Write the event that the supremum of $\{X_n\}_{n\in\mathbb{N}}$ is less than or equal to $\beta \in \mathbb{R}$, using intersections, unions, etc. to combine events of the form $[X_n \leq x]$, $[X_n \geq y]$, etc.:

$[\sup_{n<\infty} X_n \leq \beta] =$ ______________________________

b. (8) Write the event that only finitely-many X_n's exceed β (Hint: Think about $\{X_n : n \geq m\}$ for $m \in \mathbb{N}$):

$[\limsup_{n\to\infty} X_n \leq \beta] =$ ______________________________

c. (4) Why does this show that, if a sequence of $\mathcal{F}$-measurable random variables $X_n(\omega)$ converges to some value $X(\omega)$ at every $\omega \in \Omega$, then the limit $X \equiv \lim_{n\to\infty} X_n$ is also $\mathcal{F}$-measurable?

Problem 4: Let Y be a random variable taking values $0 < Y \leq 1$ and set $X \equiv Y^2$, $Z \equiv \sqrt{Y}$.

a. (10) What does Jensen's inequality tell you about the relations between the values of the expectations $a = \mathsf{E}X$, $b = \mathsf{E}Y$, and $c = \mathsf{E}Z$? Tell what convex function(s) are you using, and how.

b. (10) If Y is uniformly distributed on $(0, 1]$, compute those three expectations explicitly and verify Jensen's conclusions:

$a = \mathsf{E}X =$ ____ $\qquad b = \mathsf{E}Y =$ ____ $\qquad c = \mathsf{E}Z =$ ____

Problem 5: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the open unit interval $\Omega = (0, 1)$ with the Borel sets $\mathcal{F} = \mathcal{B}^1$ and Lebesgue measure $\mathsf{P} = \lambda$. Set $X_n(\omega) = n\,1_{(0,1/n^2]}(\omega)$ for each integer $n \in \mathbb{N}$.

a. (4) Does $X_n(\omega)$ converge at each point $\omega \in \Omega$? If so, find $X(\omega) \equiv \lim_{n\to\infty} X_n(\omega)$; if not, tell why.

Circle one: Yes No $X(\omega) =$ ____________________
Reasoning:

b. (4) Does $\int_\Omega |X_n - X|\, dP \to 0$? Y N Show why...

c. (4) Does $\int_\Omega |X_n - X|^2\, dP \to 0$? Y N Show why...

d. (4) Is $\{X_n\}$ uniformly bounded by a positive integrable RV Y? If so, find a suitable Y; if not, explain. Y N

e. (4) Is $\{X_n^2\}$ uniformly bounded by a positive integrable RV Y? If so, find a suitable Y; if not, explain. Y N

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Another Blank Worksheet