Final Examination

STA 205: Probability and Measure Theory

Due by Friday, 2006 May 5, 7:00 pm

This is an open-book take-home examination. You may work on it during any consecutive 24-hour period you like; please record your starting and ending times on the lines below.

You must do your own work—no collaboration is permitted. If a question seems ambiguous or confusing please ask me—don’t guess, and don’t discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (wolpert@stat.duke.edu).

You must show your work to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 7pm Friday, 2006 May 5. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

<table>
<thead>
<tr>
<th>Print Name:</th>
<th>1.</th>
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<tr>
<td>Issued: 4:05 , Apr 24, 2006</td>
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<td>6.</td>
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Problem 1: Let $\xi_j \overset{\text{iid}}{\sim} \text{Ex}(1)$ be standard (unit-mean) exponentially-distributed random variables; set

$$X_n = \max_{j \leq n} \xi_j.$$ 

a) Find the distribution function for $X_n$, correct for all $x \in \mathbb{R}$:

$$F_n(x) = \begin{cases} & \text{ } \\
\end{cases}$$

b) Find a sequence $a_n \to \infty$ of (non-random) real numbers for which $(X_n - a_n)$ has a non-trivial limiting distribution, and find the distribution function for the limit, correctly for all $z \in \mathbb{R}$:

$(X_n - a_n) \Rightarrow Z$, $a_n = \begin{cases} & \text{ } \\
\end{cases}$

$$F_Z(z) \equiv P[Z \leq z] = \begin{cases} & \text{ } \\
\end{cases}$$
Problem 1 (cont’d): Recall that $\xi_j \sim \text{Ex}(1)$, and set

\[ Y_n \equiv \min_{j \leq n} \xi_j. \]

c) Find the distribution function for $Y_n$, correct for all $y \in \mathbb{R}$:

\[
F_n(y) = \begin{cases} \end{cases}
\]

\[ d) \text{ Find a sequence } b_n \to \infty \text{ of positive real numbers for which } b_n \cdot Y_n \text{ has a non-trivial limiting distribution, and find the distribution function for the limit (correctly for all } z \in \mathbb{R}): \]

\[ b_n \cdot Y_n \Rightarrow Z, \quad b_n = \quad \]
Problem 2: Fix $p \in \mathbb{N}$ and let $\Omega \equiv \{1, \ldots, p\}$, with power set $\mathcal{F} \equiv 2^\Omega$, and let $P(A) \equiv |A|/p$ for $A \in \mathcal{F}$ be the uniform probability measure on $\Omega$.

a) If $p$ is prime, and if $A, B \in \mathcal{F}$ are independent (i.e., $A \perp B$), show that either $A$ or $B$ is $\Omega$ or $\emptyset$.

b) Find the joint characteristic function

$$
\phi(t_1, t_2) = \mathbb{E}[e^{it_1X_1 + it_2X_2}]
$$

for the indicator random variables $X_i(\omega) \equiv 1_i(\omega)$. Use it to confirm that $X_1$ and $X_2$ are not independent.
Problem 2 (cont’d): Recall that $\Omega \equiv \{1, \cdots, p\}$, $\mathcal{F} \equiv 2^\Omega$, and $P(A) \equiv |A|/p$ for $A \in \mathcal{F}$. Now fix any $n \in \mathbb{N}$ and let $\Omega^n = \{\bar{\omega} = (\omega_1, \cdots, \omega_n)\}$ be the Cartesian product, with product $\sigma$-field $\mathcal{F}^n$ and product measure $P^n = P \otimes P \otimes \cdots \otimes P$, describing $n$ independent uniform draws from $\Omega$. For $1 \leq i \leq j \leq n$ consider the $\binom{n}{2}$ events

$$A_{ij} \equiv \{\bar{\omega} \in \Omega^n : \omega_i = \omega_j\}$$

(c) Find $P[A_{ij}]$ correctly for each of the pairs $1 \leq i \leq j \leq n$.

d) Are the $\binom{n}{2}$ events $\{A_{ij}\}$ pairwise-independent? Prove it.

e) Are the $\binom{n}{2}$ events $\{A_{ij}\}$ independent? Prove it.
Problem 3: Let \( X, Y \overset{iid}{\sim} \text{Ex}(\lambda) \) be independent random variables, each with the exponential distribution with mean \( 1/\lambda \). Find:

a) The probability:
\[
P[X \geq 2Y] = 
\]

b) The p.d.f. for \( Z \equiv Y/X \), correctly for all \( z \in \mathbb{R} \):
\[
f_Z(z) = 
\]
Problem 3 (cont’d): Recall $X, Y \sim \text{Ex} \lambda$ and $Z \equiv Y/X$. Find:

c) The indicated expectation:

$E[Z] = \_\_\_\_\_\_\_\_\_\_$

d) Show that the characteristic function $\phi(t) \equiv E[e^{it\log(Z)}]$ for $\log(Z)$
is an even real-valued function, i.e., satisfies $\phi(-t) = \phi(t) = \overline{\phi(t)}$.

(XC: Find $\phi(t)$ explicitly):
Problem 4: Let \( \{X_n > 0\} \) and \( X > 0 \) be positive random variables with \( X_n \to X \) a.s. Choose True or False below, and sketch a proof (e.g., cite a relevant theorem) or give a counter-example to show that you’re right.

a)  True or False \( \log(X_n) \to \log(X) \) in probability.

b)  True or False \( X_n \to X \) in \( L_2 \) if each \( \mathbb{E}[|X_n|^3] \leq \pi \).

c)  True or False \( \log(X_n) \to \log(X) \) in \( L_1 \) if each \( \mathbb{E}[|X_n|^3] \leq \pi \).
Problem 4 (cont’d): Recall \( \{X_n > 0\} \), \( X > 0 \), and \( X_n \rightarrow X \) a.s.

\[ \text{d) } T \ F \quad \limsup_{n \to \infty} E[\log(1 + X_n)] \geq E[\log(1 + X)] \]

\[ \text{e) } T \ F \quad X \in L_2 \text{ if } E[\exp(t \cdot X)] < \infty \text{ for some } t > 0. \]

\[ \text{f) } T \ F \quad X \in L_2 \text{ if } E[\exp(t \cdot X)] < \infty \text{ for some } t < 0. \]
Problem 5: Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathbb{B}_1, \lambda)$ be the unit interval with Lebesgue measure $\lambda(d\omega) = d\omega$ (length). Define three random variables by

$$X(\omega) \equiv 1_{[0,5]}(\omega) \quad Y(\omega) \equiv 1_{(3,7]}(\omega) \quad Z(\omega) \equiv \omega(1-\omega).$$

a) For each pair of $\sigma$-algebras below, indicate with a symbol on the line whether there is inclusion ($\subset$ or $\supset$), equality ($\equiv$), independence ($\perp\perp$), or none of these ($\times$):

$$\sigma(X) \quad \sigma(X) \quad \sigma(Y)$$
$$\sigma(X) \quad \sigma(Z)$$
$$\sigma(Y) \quad \sigma(Z)$$
$$\sigma(X,Y) \quad \sigma(Z)$$
$$\sigma(X,Z) \quad \mathcal{F}$$

b) Find and plot the conditional expectation $E[Z|X,Y]$:

\[0.0 \quad 0.5 \quad 1.0\]
\[0.0 \quad 0.5 \quad 1.0\]
Problem 5 (cont’ed): Recall $(\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}_1, \lambda)$ and

$$X(\omega) \equiv 1_{[0,0.5]}(\omega) \quad Y(\omega) \equiv 1_{[3,0.7]}(\omega) \quad Z(\omega) \equiv \omega(1 - \omega).$$

Define a new random variable by $W(\omega) \equiv 1_{[0,2,0,6]}(\omega)$

c) Find and plot the conditional expectation $E[W|Y]$:

d) Find and plot the conditional expectation $E[W|Z]$:  

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**Problem 6:** Let $\xi_j \sim \text{Un}(0, 2)$ be independent random variables with the uniform distribution on the interval $(0, 2)$, and set $X_n \equiv \prod_{j=1}^{n} \xi_j$.

a) Find the mean and variance of $X_n$:

$E[X_n] = \quad V[X_n] = \quad$

b) For each $p > 0$, can you find a number $a_p > 0$ such that

$M_p(n) \equiv (a_p)^n \cdot (X_n)^p$

is a martingale?

c) Bound as tightly as you can

$P[X_{16} > 0.010] \leq \quad$

d) Bound as tightly as you can

$P \left[ \sup_{0 \leq n \leq 16} X_n > 2.0 \right] \leq \quad$

e) (XC) For $r, \lambda > 0$ find bounds for $P[\sup_{n<\infty} r^n X_n > \lambda]$
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p q^{x-1}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = p q^{y-1}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$nP$</td>
<td>$nP (1-P) \frac{n-x}{n}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta [1+e^{-(x-\mu)/\beta}]^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 / 3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(\log(x-\mu))^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2} (1 - e^{-\sigma^2})$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, \beta)$</td>
<td>$f(x) = \binom{x+\alpha-1}{x} p^\alpha q^{x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = \binom{y-1}{y-\alpha} p^\alpha q^{y-\alpha}$</td>
<td>$y \in {\alpha, \ldots}$</td>
<td>$\alpha/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \beta)$</td>
<td>$f(x) = \beta \alpha^\beta x^{\alpha-1} \beta^{-\alpha}$</td>
<td>$x \in (\alpha, \infty)$</td>
<td>$\frac{\alpha \beta}{\beta-1}$</td>
<td>$\frac{\sigma^2 \beta}{(\beta-1)^2(\beta-2)}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor</td>
<td>$\text{F}(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} x^{\frac{\nu_1}{2}-1} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_1}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_1}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}} [1 + x^2/\nu]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu/\nu$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta, \gamma)$</td>
<td>$f(x) = \frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha} e^{-((x-\gamma)/\beta)\alpha}$</td>
<td>$x \in (\gamma, \infty)$</td>
<td>$\gamma + \beta \Gamma(1+\alpha^{-1})$</td>
<td></td>
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</tbody>
</table>