Biometrika (1987), **74**, 3, *pp.* 646–8
Printed in Great Britain

On scale mixtures of normal distributions

BY MIKE WEST
Department of Statistics, University of Warwick, Coventry CV4 7AL, U.K.

SUMMARY

The exponential power family of distributions of Box & Tiao (1973) is shown to be a subset of the class of scale mixtures of normals. The corresponding mixing distributions are explicitly obtained, identifying a close relationship between the exponential power family and a further class of normal scale mixtures, namely the stable distributions.

Some key words: Exponential power family; Scale mixture of normals; Stable distribution.

1. SCALE MIXTURES OF NORMALS

Suppose that Y has a standard normal distribution and that σ has some distribution on $(0, \infty)$ with a continuous or discrete density $h(\sigma)$ $(\sigma > 0)$. Then the distribution of $X = Y\sigma$ is referred to as a scale mixture of normals, and with a scale mixing density $h(\sigma)$. A wide class of continuous, unimodal and symmetric distributions on the real line may be constructed as scale mixtures of normals. Many examples, such as discrete mixtures or contaminated normals, the Student t family, logistic, Laplace or double-exponential, and the stable family, are well known; see, for example, Andrews & Mallows (1974) and West (1984). Their properties have been useful in several areas. In theoretical studies distributional properties such as moments are often easily derived by exploiting the special structure. In practice, robustness studies have often used these distributions for simulation and in the analysis of outlier models; see West (1984) and references therein.

In this note, a new family of normal scale mixtures is identified. This class of distributions, the exponential power family, has been used widely in robustness studies; it was introduced and popularized by Box & Tiao (1973) in the context of Bayesian modelling for robustness. However, the normal scale mixture property and an interesting relationship with the class of stable distributions have not, so far, been discussed. That exponential power distributions are normal scale mixtures may be proved in two ways. The first method uses the characterization result of Andrews & Mallows (1974). The second employs a direct method that explicitly identifies the scale mixing distribution.

The characterization result referred to above is as follows. Suppose that X is a real-valued random quantity with a continuous, unimodal and symmetric distribution having density $p(X)$ $(-\infty < X < \infty)$. Without loss of generality, suppose that the mode is at zero. Then the symmetry assumption implies that $p(X) = f(\frac{1}{2}X^2)$ for some positive and decreasing function $f(U)$ $(U \geqslant 0)$. Chu (1973) shows that $p(.)$ has the form

$$p(X) = \int_0^\infty \sigma^{-1}\phi(\sigma^{-1}X)h(\sigma)\, d\sigma, \tag{1}$$

where $\phi(.)$ is the standard normal density and $h(.)$ is some function on $(0, \infty)$. Andrews & Mallows (1974) characterize the class of normal scale mixtures on the real line by proving that $h(.)$ is a density function if and only if $(-1)^n D^n f(U) \geqslant 0$ $(U \geqslant 0)$, for each positive integer n, where $D^n f(U)$ is the nth derivative of $f(.)$ at U. Note that there are densities $p(.)$ for which $h(\sigma)$ is not a density. Chu (1973) provides the example $p(X) \propto 1/(4+X^4)$ for which $h(\sigma) \propto \sigma^{-2} \sin(\sigma^{-2})$ is not even nonnegative everywhere on $(0, \infty)$.

Example: The symmetric stable family. The real-valued random quantity X has a symmetric stable distribution with index a $(1 \leqslant a \leqslant 2)$, location 0 and scale 1, if and only if the characteristic function of X is

$$\Psi_X(t) = E(e^{itX}) = \exp(-|t|^a) \quad (-\infty < t < \infty). \tag{2}$$

The endpoints of this class are given by $a = 1$, the Cauchy distribution, and $a = 2$, the normal. It is well known that, for all a in $(1, 2)$, the density $p(X)$ of X has the form in (1). In these cases, $h(\sigma)$ is the density of σ when σ^2 has a positive stable distribution with index $\frac{1}{2}a$ (Feller, 1966, § 6.2, p. 172).

2. The exponential power family

The standard exponential power family (Box & Tiao, 1973, § 3.2.1, p. 156) is comprised of all those distributions, having location 0 and scale 1, with densities of the form

$$p(X) = k \exp(-|X|^b) \quad (-\infty < X < \infty), \tag{3}$$

for some b $(1 \leq b \leq 2)$. The distribution is also defined, of course, for $b > 2$ although, in such cases, cannot be represented as a scale mixture of normals. Two important special cases are the normal $(b = 2)$, and the Laplace or double-exponential $(b = 1)$, both of which are known to be scale mixtures of normals, the former being a degenerate mixture. The fact that this is true for all b $(1 \leq b \leq 2)$ is easily verified using the results of Andrews & Mallows (1974) mentioned earlier, although this approach does not identify the mixing distribution. This may be done as follows.

Theorem. *The density* (3) *has the form* (1) *with mixing distribution for* σ *having density* σ^{-2} $p_{\frac{1}{2}b}(\sigma^{-2})$ $(\sigma > 0)$, *where* $p_a(.)$ *is the density of the positive stable distribution of index* a $(0 < a < 1)$.

Proof. The density function (3) is proportional to the characteristic function (2), $\Psi_U(X) = \exp(-|X|^b)$ of the stable distribution for U of index b with location 0 and scale 1. Now if $q(U)$ denotes the density of such a distribution, then, by definition,

$$p(X) = k\Psi_U(X) = k \int_{-\infty}^{\infty} e^{iUX} q(U)\, dU.$$

Now, from the example in § 1, $q(U)$ may be written in the form (1),

$$q(U) = \int_0^{\infty} \sigma^{-1} \phi(\sigma^{-1} U) h(\sigma)\, d\sigma,$$

where $h(\sigma)$ is the density of σ when σ^2 is positive stable of index $\frac{1}{2}b$. Thus

$$p(X) = k \int_{-\infty}^{\infty} e^{iUX} \left\{ \int_0^{\infty} \sigma^{-1} \phi(\sigma^{-1} U) h(\sigma)\, d\sigma \right\} dU.$$

Noting that $\int e^{iUX} \phi(\sigma^{-1} U)\, dU < \infty$, where the integral is over $(-\infty, \infty)$, for all X to justify the interchange of the orders of integration here, we have

$$p(X) = k \int_0^{\infty} h(\sigma) \left\{ \int_{-\infty}^{\infty} e^{iUX} \sigma^{-1} \phi(\sigma^{-1} U)\, dU \right\} d\sigma.$$

But the term in braces here is simply the characteristic function at a point X of the zero-mean normal distribution with variance σ^2, and so is given by $\exp(-\frac{1}{2}\sigma^2 X^2)$. Thus $p(X) = k \int h(\sigma) \exp(-\frac{1}{2}\sigma^2 X^2)\, d\sigma$, where the integral is over $(0, \infty)$. Transformation to $\tau = \sigma^{-1}$ in the integrand leads to

$$p(X) = k \int_0^{\infty} \tau^{-2} h(\tau^{-1}) \exp(-\tfrac{1}{2} X^2/\tau^2)\, d\tau = k \int_0^{\infty} g(\tau) \tau^{-1} \phi(\tau^{-1} X)\, d\tau,$$

where $g(\tau) \propto \tau^{-1} h(\tau^{-1})$ $(\tau > 0)$. Hence $p(.)$ has the form in (1) with mixing density $g(.)$. Finally note that $g(\tau) \propto \tau^{-2} p_{\frac{1}{2}b}(\tau^{-2})$ $(\tau > 0)$, where $p_a(.)$ is the density of the stable distribution of index a $(0 < a < 1)$. □

When $b = 1$, X has a double-exponential distribution and, following Feller (1966, § 6.2, p. 170), $p_{\frac{1}{2}}(\sigma) \propto \sigma^{-3} \exp(-\frac{1}{2}/\sigma)$ $(\sigma > 0)$, so that $g(\tau) \propto \tau \exp(-\frac{1}{2}\tau^2)$, or, if $\lambda = \sigma^2$, then λ has an exponential distribution with mean 2. This deduction from the general result of Theorem 2 agrees with that derived for this special case of Andrews & Mallows (1974).

REFERENCES

ANDREWS, D. R. & MALLOWS, C. L. (1974). Scale mixtures of normal distributions. *J. R. Statist. Soc.* B **36**, 99-102.

BOX, G. E. P. & TIAO, G. C. (1973). *Bayesian Inference in Statistical Analysis.* Reading, Mass: Addison-Wesley.

CHU, K. C. (1973). Estimation and detection in linear systems with elliptical errors. *IEEE Trans. Auto. Control* **18**, 499-505.

FELLER, W. (1966). *An Introduction to Probability Theory and its Applications*, **2**. New York: Wiley.

WEST, M. (1984). Outlier models and prior distributions in Bayesian linear regression. *J. R. Statist. Soc.* B **46**, 431-9.

[*Received December* 1984. *Revised September* 1986]