I. Random variables.

(A) Let $(\Omega, \mathcal{B}, P) = ((0, 1], \mathcal{B}((0, 1]), \lambda)$ for Lebesgue measure $\lambda$. Define:

\[
\begin{align*}
X_1(\omega) &\equiv 0, \quad \forall \omega \in \Omega \\
X_2(\omega) &\equiv 1_{\{1/2\}}(\omega) \\
X_3(\omega) &\equiv 1_\mathbb{Q}(\omega)
\end{align*}
\]

where $\mathbb{Q}$ is set of rational numbers in $(0, 1]$. Let $f(k) \equiv P[X_1 = X_2 = X_3 = k]$, $k \in \mathbb{R}$. Plot the function $f(\cdot)$. What are the $\sigma$-algebra’s generated by $X_1, X_2$ and $X_3$?

(B) Toss two independent fair coins and define a random variable $\zeta$ as follows: If both the coins show Heads, $\zeta = 1$; if both the coins show Tails $\zeta = 2$; and if the two coins differ, $\zeta = 0$. Suggest a suitable probability space $(\Omega, \mathcal{F}, P)$ for this problem—specify a sample space $\Omega$, write $\zeta$ explicitly as a function on $\Omega$, describe the $\sigma$-algebra $\mathcal{F}$ generated by $\zeta$, and give the probability assignment on $\mathcal{F}$ for fair, independent coins. Is $\mathcal{F}$ identical to the power set $\mathcal{P} := \{A : A \subset \Omega\}$ of $\Omega$? Is $\zeta$ measurable with respect to $\mathcal{P}$?

II. More on Random variables.

(A) Let $X$ be a random variable with CDF $F(x) := P(X \leq x)$. Set $Y \equiv F(X)$. If $X$ has a continuous distribution, show that $Y$ is a random variable and that $Y$ has a uniform distribution on $[0, 1]$.

(B) If $X$ is a real valued random variable (so $P[|X| < \infty] = 1$), then show that for any $\epsilon > 0$, there exists a bounded random variable $Y$ such that

\[P(X \neq Y) < \epsilon\]

(A random variable $Y$ is bounded if it satisfies $|Y(\omega)| \leq K_\epsilon$ for some fixed number $K_\epsilon < \infty$ that does not depend on $\omega$.)

III. Measurable functions.

(A) Let $\Omega = \mathbb{R}$, and set $\mathcal{S} = \{\Omega, \emptyset, (-\infty, 0], (0, \infty)\}$. Show that $\mathcal{S}$ is a $\sigma$-algebra. What functions $f$ defined on $\Omega$ are $\mathcal{S}$ measurable?

(B) If $X$ is a random variable, then show that so is $|X|$. Show by an example that the converse need not be true.
(C) Let \( \Omega = \mathbb{R} \). Let \( \mathcal{S}_0 \equiv \{\emptyset, \Omega\} \) be the so-called trivial \( \sigma \)-algebra. Consider the function \( X(\omega) = \omega^2 : \Omega \to \mathbb{R} \). Is the function \( X \mid_{\mathcal{S}_0} \) measurable? Justify your answer. Find the \( \sigma \)-algebra \( \mathcal{F}_X := \sigma(X) \) generated by \( X \). Is the set \((-\infty, 0]\) in \( \mathcal{F}_X \)? How about \([-4, 4] \)?

(D) Let \( \{X_n, n \geq 0\} \) be real-valued random variables on the probability space \( (\Omega, \mathcal{B}, P) \) that satisfy
\[
\lim_{n \to \infty} X_n(\omega) = +\infty
\]
for every \( \omega \in \Omega \), and let \( B < \infty \) be a real number. Consider the integer-valued quantities
\[
\tau(\omega) \equiv \inf\{n \geq 0 : X_n(\omega) \geq B\}
\]
Prove that \( \tau \) is a random variable.

**Extra credit:** Prove that \( X_\tau \) is also a random variable.

IV. **Practice with limsup and liminf.**

(A) Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of random variables on \( \Omega \) and set \( Y \equiv \limsup X_n \). For \( \beta \in \mathbb{R} \), express the event \( \{\omega \in \Omega : Y(\omega) \leq \beta\} \) in terms of unions, intersections, and complements of events of the form \( \{\omega \in \Omega : X_n(\omega) \leq \beta\} \). Why is \( Y \) a random variable?

(B) Now set \( Z \equiv \liminf X_n \). For \( \alpha \in \mathbb{R} \), express the event \( \{\omega \in \Omega : Z(\omega) \leq \alpha\} \) in terms of unions, and intersections, and complements of events of the form \( \{\omega \in \Omega : X_n(\omega) \leq \alpha\} \). Why is \( Z \) a random variable?