Sta 205 : Homework #5
Due : February 21, 2007

1. Independence.

(a) Let \{B_i\}, be independent events. For \(N \in \mathbb{N}\) show that
\[
P \left( \bigcup_{i=1}^{N} B_i \right) = 1 - \prod_{i=1}^{N} [1 - P(B_i)]
\]

(b) If \(\{A_n, n \in \mathbb{N}\}\) is a sequence of events such that
\[
P(A_n \cap A_m) = P(A_n)P(A_m) \forall n, m \in \mathbb{N}, n \neq m,
\]
does it follow that the events \(\{A_n\}\) are independent? Give a proof or counter-example.

(c) Let \(X\) be a random variable. Show that \(X\) is independent of itself if and only if there is some constant \(c \in \mathbb{R}\) for which \(P[X = c] = 1\). Let \(f\) be a Borel measurable function, and \(X\) a (not necessarily constant) random variable. Can \(f(X)\) and \(X\) be independent? Explain your answer.

(d) Show that if the event \(A\) is independent of the \(\pi\)-system \(\mathcal{P}\) and \(A \in \sigma(\mathcal{P})\), then \(P(A)\) is either 0 or 1.

(e) Give a simple example to show that two random variables on the same space \((\Omega, \mathcal{F})\) may be independent according to one probability measure \(P_1\) but dependent with respect to another \(P_2\).

2. Practice with Borel Cantelli.

(a) Let \(\{X_n\}\) be a sequence of Bernoulli random variables with
\[
P(X_n = 1) = n^{-p} \quad P(X_n = 0) = 1 - n^{-p}
\]
for some \(p > 0\). For \(p = 2\) show that the partial sum
\[
S_n := \sum_{k=1}^{n} X_k
\]
converges almost-surely, whether or not the \(\{X_n\}\) are independent. If the \(\{X_n\}\) are independent, with \(p = 1\), does \(S_n\) converge? Why or why not?
(b) Dane tosses a heavily biased coin repeatedly, with independent outcomes. He is convinced that if he chooses the probability of heads \( p \) to be small enough (say, \( p \approx 10^{-6} \)), then only finitely-many heads will ever appear. Is Dane right? Justify your answer.

(c) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Let \( \{X_n\} \) be a sequence of i.i.d. non-trivial (i.e., not almost-surely constant) random variables, then show that

\[
P[X_n \text{ converges}] = 0
\]

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables \( \{X_n\} \), there exists constants \( c_n \to \infty \) such that

\[
P \left( \lim_{n \to \infty} \frac{X_n}{c_n} = 0 \right) = 1.
\]

Give a careful description of how you choose \( c_n \). Find a suitable sequence \( \{c_n\} \) explicitly for an i.i.d. sequence \( \{X_n\} \) iid \( \mathcal{N}(0,1) \) of standard Gaussian random variables to ensure that \( X_n/c_n \to 0 \) almost surely.


(a) Suppose \( \{A_n, n \in \mathbb{N}\} \) are independent events satisfying \( P(A_n) < 1, \forall n \in \mathbb{N} \). Show that \( P(\bigcup_{n=1}^{\infty} A_n) = 1 \) if and only if \( P(A_n \text{ i.o.}) = 1 \). Give an example to show that the condition \( P(A_n) < 1 \) cannot be dropped.

(b) Suppose \( \{A_n\} \) is a sequence of events. If \( P(A_n) \to 1 \) as \( n \to \infty \), prove that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\bigcap_{k} A_{n_k}) > 0 \).

(c) Let \( A_n \) be a sequence of events. If there exists \( \epsilon > 0 \) such that \( P(A_n) \geq \epsilon \) for all \( n \in \mathbb{N} \), does it follow that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\bigcap_{k} A_{n_k}) > 0 \)? Why or why not?

(d) Let \( \{X_n\} \) be i.i.d. random variables, with tail \( \sigma \)-field

\[
\mathcal{T} \equiv \bigcap_n \mathcal{F}_n, \quad \mathcal{F}_n \equiv \sigma\{X_m : m \geq n\}
\]

Is the event

\[
E = \{\text{There exists a number } B < \infty \text{ such that } |X_n| \leq B \text{ for infinitely-many } n\}
\]

\[
= \bigcup_{B=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{\omega : |X_n(\omega)| \leq B\}
\]

in \( \mathcal{T} \)? Prove or disprove it. Find the probability \( P[E] \) in terms of the random variables’ common CDF, \( F(x) \equiv P[X_n \leq x] \).